Fertility in an Unequal, Innovative World*

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Abstract

We show how inequality, operating through the returns to knowledge creation driven by technological change, has shaped the *income-fertility* relationship from the past to the present, and will influence its future trajectory. Our model also replicates historical fertility dynamics through to the present and projects a spectrum of future paths shaped by inequality-technological change dynamics including, in the future, the prospect of an Empty Planet (**EP**) scenario, as highlighted in Jones (2022), or a renewed drift toward Malthusian-like (**ML**) stagnation.

Keywords: Fertility; Growth; Technological change; Inequality
JEL codes: D1, J1, O10

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1 Introduction

Technological progress shapes fertility decisions by altering returns to education, which affects both educational investment and income inequality across groups. This paper argues that such heterogeneity in the interaction between technology, human capital, and fertility jointly explains the historical *income-fertility* relationship and the aggregate fertility transition over time, tracing the evolution from the Malthusian era to modern times and projecting future trends in *income-fertility* dynamics. We bridge a key gap in the literature by providing a unified framework that connects these mechanisms, emphasizing that (a) returns to education are inherently linked to technological progress, and (b) inequality driven by technological progress *via* returns to education remains fundamental. In doing so, we build upon and critically reassess seminal contributions that, while illuminating important dimensions of this nexus, fall short of capturing the full dynamic interplay we uncover.

We elaborate the key mechanism that sets our paper apart. We focus solely on inequality and its root, technological growth both of which empirically measurable. We define potential income as a combination of a fixed income, equal for all individuals, and a variable component linked to relative productivity in line with de la Croix and Doepke (2003). In the initial years, since the cost of educating a child is fixed and independent of parental productivity, an increase in fixed income raises potential income and cost of education. This assumption is crucial in explaining fertility differentials: higher fixed income incentivizes parents to have more children but invest less in their education. Conversely, as relative productivity rises, the time cost of raising multiple children increases, making it optimal for highly productive parents to prioritize education over family size. This transition marks the shift from the Malthusian regime - where fertility increases with potential income - to the Modern-Growth (**MG**) regime, where rising potential income leads to lower fertility due to higher opportunity costs. In some regions, however, fertility may begin to rise again with potential income, when the effect of higher fixed income outweighs opportunity costs. We term this phase the More-Modern-Growth (**MMG**) regime.

With technological advancements, the returns to education increase as in Galor and Weil (2000) and Galor and Moav (2002) - technology complements skills in the production of human capital and as a result, parents prioritize investing in education over having more children. Moreover, as education becomes more rewarding, wealthier individuals allocate more resources to their children's education than poorer individuals. Consequently, the children of affluent families become relatively more productive, widening the gap between rich and poor and driving greater inequality. The relationship between inequality and fertility depends on the prevailing economic regime. In the Malthusian and **MMG** regimes, rising inequality leads to higher overall fertility. This occurs because wealthier individuals, benefiting from increased income disparities, choose to have more children, while poorer individuals slightly reduce their family size. In contrast, under the **MG** regime, greater inequality results in lower overall fertility, as wealthier individuals opt to have fewer children compared to those with lower incomes. Thus, the historical trajectory of fertility can be explained by the relative influence of technological progress - whether it primarily enhances education or exacerbates inequality through rising returns to education.

Increased investment in parents' education raises their fixed income, expanding fertility choices (income effect). It also drives technological progress in the next generation, as progress depends on population size and parental education. As in Galor and Weil (2000), this increases returns on children's education, reducing fertility (substitution effect). However, inequality rises, as wealthier parents invest more in their children's education, making their offspring relatively more productive. Technological progress and its effect on inequality shape an economy's escape from the **Malthusian** trap. In the **MG** regime, technological progress reduces fertility, as the substitution effect outweighs the income effect, increasing overall education. Rising inequality further decreases fertility and boosts educational investment, potentially leading to an Empty Planet (**EP**) scenario as in Jones (2022) if population decline outweighs educational gains. In the **MMG** regime, technological progress leads to an increase in fertility while lowering education as income effect dominates substitution effect. Here, inequality drives both fertility and education investment, potentially leading to sustained output growth. Importantly, this is not the sole possible outcome; for instance, a resurgence of Malthusian-like **(ML)** stagnation, characterized by positive population growth, may also arise in the **MMG** era. Thus, ultimately, technological progress and its impact on inequality determine an economy's long-term trajectory.

The literature on long-run growth and demographic transition is rich yet fragmented. Studies such as Galor and Weil (2000) Galor and Moav (2002) develop unified growth frameworks capturing fertility dynamics. Others - including Jones (2001), Kogel and Prskawetz (2001), Hansen and Prescott (2002), Tamura (2002), and Doepke (2004) - analyze the transition from pre-industrial stagnation to sustained growth, often highlighting demographic shifts. However, these studies largely abstract from inequality and do not incorporate recent important developments in fertility trends or long-term forecasts. In contrast, de la Croix and Doepke (2003), building on Glomm and Ravikumar (2003) and Becker and Barro (1998), explicitly model inequality and differential mortality for some limited period, yet omit the role of technological progress in shaping human capital returns. Moav (2005) develops a theory where families jointly choose fertility and child education, with higher human capital enhancing individuals' effectiveness as teachers. It explains persistent poverty across and within countries and shows that inequality hampers growth both through its impact on fertility decisions and by reducing the relative returns to human and physical capital. More recently, Jones (2022) highlights the unintended macroeconomic consequences of population decline, including the stagnation of living standards. Taken together, some of these contributions may appear contradictory when viewed under a unified framework, due to varying assumptions and temporal scopes. We show that the stagnation outcome emphasized in the *Empty Planet* **(EP)** narrative - where negative population growth leads to zero economic growth - can arise endogenously, depending on the relative strength of inequality and technological progress.

Several other studies also contribute important perspectives. Demographic transition, mortality, longevity, human capital, and growth have also been studied together by Boucekkine et. al. (2002, 2003), de la Croix and Licandro (2013) among others. Cordoba and Ripoll (2014) observes that *income-fertility* relationships vanish if parents could legally impose debt on children to recover upbringing costs, but emerge when such constraints exist, turning negative when intergenerational elasticity of substitution exceeds unity (also see altruistic agents setups in Cordoba and Ripoll (2019) and Cordoba et al. (2016)). Bhattacharya and Chakraborty (2007) links fertility behavior to child mortality and contraception, Aksan and Chakraborty (2014) presents demographic, disease and economic transition in a general equilibrium. Finally, our findings can also explain the recent observation by Doepke et al. (2023) that the fertility-income relationship has exhibited a slight upward trend. In our view, while they provide valuable insights into rising fertility, child care services likely concentrated among the wealthy among others, they do not examine how technological shocks, mediated by education returns and affordability, create inequality that shapes the historical and evolving *income-fertility* relationship, fertility over time or its future path.

The rest of the paper is organized as follows. Section 2 discusses some important studies that are relevant for the present analysis, explains why these studies differ in their outcomes and points out the discomfort. In Section 3, we present our comprehensive model. Section 4 explicitly incorporates inequality in the setup. Section 5 provides empirical support to our findings. While Section 6 represents the global dynamics and various possibilities of economies in the long run, Section 7 provides computational outcome and Section 8 concludes.

2 Also some discomfort in reconciling

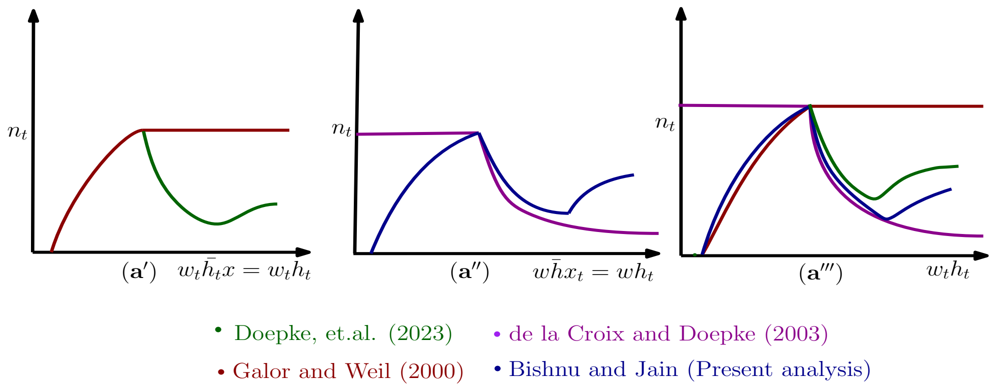
As briefly mentioned above, there is also a discomfort in reconciling some of the existing studies to explain *income-fertility* relationship. Galor and Weil (2000) which assumes no within cohort inequality, poor households show a positive link between income and fertility, while when they are rich, no such relationship is observed (red,

Figure 1(a′)). In contrast, de la Croix and Doepke (2003) where within cohort inequality is present, argue that poor households lack any income-fertility link, whereas rich households exhibit a negative relationship (magenta, Figure 1(a″)). Doepke et al. (2023) highlight a shift in high-income countries where the income-fertility relationship has turned positive (in the **MMG** regime as per our notation) due to the marketization of childcare services. With marketized childcare, the cost of raising children moves from opportunity costs to monetary costs, making the income-fertility relationship positive for high-income countries (green, Figure 1(a′)). While we do not provide direct confirmation of these recent shifts[1], we can assert that, if such shifts are indeed occurring, inequality and technological progress together can account for a positive *income–fertility* relationship in the **MG** regime, even without explicitly incorporating childcare into the model.[2] Alongside these studies, we present our own findings (labeled as Bishnu and Jain (Present analysis), blue line, Figure 1(a‴)), which are broadly consistent with the empirically observed *income–fertility* pattern in most OECD countries. While Figure 1(b) shows the *income-fertility* trajectories for the USA, Canada, Australia, and New Zealand, Appendix B.5 presents the corresponding data for all 21 OECD countries.

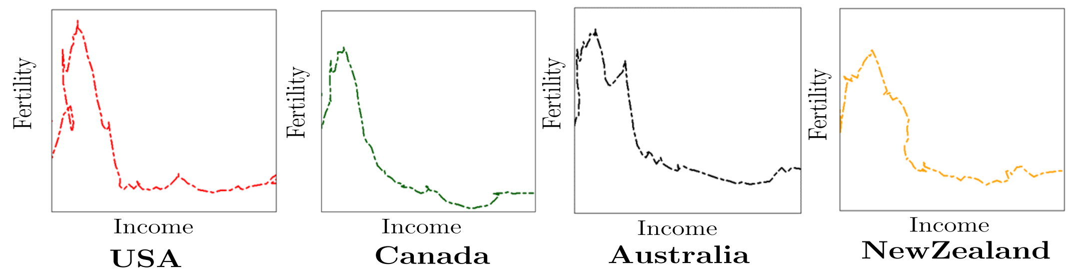
Related to the previous one, there is another challenge that lies in reconciling the existing studies to provide a consolidated explanation for the factors that drive growth. In Galor and Weil (2000), technological progress implicitly drives parental investment in children's education. However, during the Malthusian regime, technological progress predominantly leads to an increase in population size, as the income effect outweighs the substitution effect. Consequently, this population growth enables an economy to take off, eventually transitioning into the **MG** regime. In this regime, fertility is negatively related to investment in a child's education. The cumulative effect of technological progress leads to a decline in fertility, as income no longer influences fertility decisions. Output per capita rises as technological progress outpaces population growth. In contrast to that, in de la Croix and Doepke (2003) where differential level of relative human capital makes households heterogeneous in terms of potential income, output falls as differential fertility rises with technological progress. Increasing differential fertility lowers weighted average education as greater weight is given to families with lower education levels as poor households prioritize child quantity over quality. Due to the assumption that technological progress complements skills in the production of human capital, implies that the parameter η, which determines returns to

[1] This implies that our comprehensive model can generate a perpetually declining *income-fertility* trajectory, depending on how the relative effects unfold, which in turn is governed by economy specific parameter values.

[2] This naturally confirms that explicit inclusion of childcare in an unequal society, can definitely generate the observed slightly increasing trend in the **MG** regime. We have not presented the model with the availability of child care services at the **MG** regime that can confirm the findings of Doepke et al. (2023) because of the space constraint, however, all the results are available on request.



(a) **Income-fertility path** based on Models.



(b) **Income-fertility path** based on Data.

Figure 1: **Income-fertility** relationship.

education and differential fertility in de la Croix and Doepke (2003), increases. Therefore, when consolidating the two theories, we observe contrasting outcomes: according to Galor and Weil (2000), technological progress leads to higher output, whereas in de la Croix and Doepke (2003), it results in higher differential fertility, ultimately causing a decrease in output. Very recently, Jones (2022) claimed that when population growth turns negative, both endogenous and semi-endogenous growth models generate *Empty Planet* **(EP)** outcome, that is, knowledge and the living standards stagnate. Thus, with technological progress, while in Galor and Weil (2000) output rises, according to Jones (2022) with fall in total fertility, over time growth rate of population turns negative and output stagnates.

There are a few more crucial distinctions that can be noticed among the studies available. For example, de la Croix and Doepke (2003) assumed that human capital is solely a function of investment in a child's education, excluding the role of technological progress in shaping returns to education and human capital. In contrast, Galor and Weil (2000) emphasizes that technological progress plays a crucial role in amplifying both the returns to education and the accumulation of human capital. As a result, in their model, economies with declining inequality, population, and rising education levels avoid an **EP** outcome. We find that the critical assumptions in Galor and Weil (2000) can be seamlessly integrated into de la Croix and Doepke (2003), which predominantly addresses inequality. For instance, we develop a model that incorporates technological progress within the framework of de la Croix and Doepke (2003), ensuring it aligns with all the standard assumptions regarding technological progress outlined in Galor and Weil (2000). After incorporating the effects of technological progress, if the decline in population growth outweighs the benefits of rising education, the resulting decrease in technological progress and η adversely affects human capital and economic growth, despite the reduction in inequality. This can lead to an **EP** outcome, even in the presence of declining inequality and population growth. We, therefore, conclude that these influential studies on endogenous fertility in the presence of technological progress under common and comparable assumptions generate differential outcome. That is, with technological progress output increases in Galor and Weil (2000), decreases in de la Croix and Doepke (2003) and stays constant in Jones (2022). Our comprehensive model that captures the entire historical and possible future paths is also capable of explaining the source of core differences in the outcome of these existing papers.

We now turn to the discomfort in explaining the historical fertility paths of countries, that is fertility transitions over time too when we compare different studies. For example, according to the mechanism proposed by Galor and Weil (2000), the historical fertility path after the Malthusian regime should show a declining trend, as income plays no role in determining fertility during that regime, while fertility decreases with technological advancement. In contrast, the mechanism outlined by de la Croix and Doepke (2003) suggests that the historical fertility path depends on changes in inequality - if inequality decreases over time, the historical fertility path will display a declining trend.

Figure 2 below illustrates fertility trends *over time* across Australia, Canada, New Zealand, and the United States.[3] In all cases, we observe an initial rise in fertility followed by a decline and, subsequently, a slight increase. Between 1930 and 1950, inequality rises and fertility falls in Canada and Australia, while in the United States and New Zealand,fertility falls with fall in technological progress. Additionally, there are significant country-specific variations in fertility patterns over time.[4] To avoid confusion, we clarify that the analysis of *fertility over time* appears only in **Section 5**, whereas **Section 3** and **Section 4** focus exclusively on the *income–fertility* relationship. Thus, briefly, our reliance on inequality arising from technological progress proves advantageous, as together they can replicate all observed fertility patterns, depending on the extent of inequality's influence.

[3]Sources: Children ever born by year of birth in Allied countries. Data provided by Jean-Paul Sardon of the Observatoire Demographique Europeen, Doepke et al. (2023), and Madsen and Strulik (2023).

[4]Historical fertility paths for 21 OECD countries are presented in **Appendix B.6**.

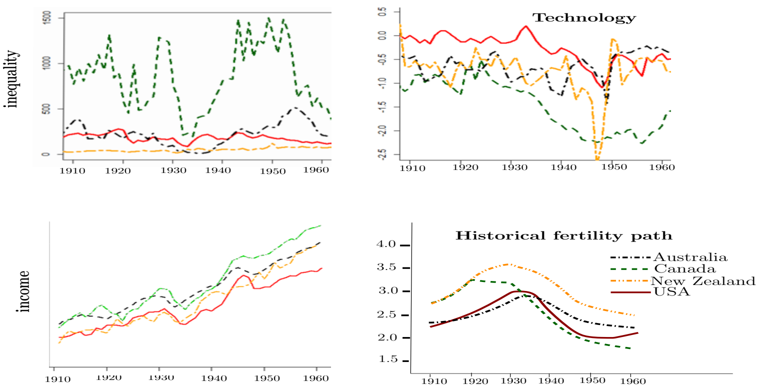


Figure 2: Inequality, Technology, Income and Fertility path **over time**.

3 Our comprehensive model

3.1 Production of final output

Production follows a constant-returns-to-scale technology that is subject to endogenous technological progress. Precisely, the final output produced at time t, Y_t, is given by

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha},$$

where K_t is aggregate capital, L_t is aggregate labor supply, $A_t > 0$ represents the endogenously determined technological progress at time t, and the parameter $\alpha \in (0, 1)$. Physical capital completely depreciates in one period. The firm chooses inputs by maximizing profits $\Pi_t = Y_t - w_t(A_t L_t) - R_t K_t$ where w_t and R_t represent wage rate and gross rate of interest on capital respectively. Factor prices follow from the assumed competitive setup of firms, which leads to equalization of marginal costs and productivities:

$$w_t = (1-\alpha)\left(\frac{K_t}{A_t L_t}\right)^{\alpha} \equiv (1-\alpha)\kappa_t^{\alpha},$$

$$R_t = \alpha\left(\frac{K_t}{A_t L_t}\right)^{\alpha-1} \equiv \alpha\kappa_t^{\alpha-1},$$

where $\kappa_t = \frac{K_t}{A_t L_t}$ is defined as capital per effective labor at time t.[5]

3.2 Preferences and budget constraints

Member of generation t live for three periods: childhood, adulthood and old age. Time is discrete and goes from 0 to ∞. In the first period of life in period $t-1$, individual consumes a fraction of their parent's time. All the decisions taken by the

[5]Krueger and Ludwig (2007) finds the effect of demographic change including rate of fertility for rates of returns to capital, and the distribution of wealth and welfare in a multi-country large-scale overlapping generations model.

individuals are in period t when they are adult in the second period of life. While adult, households are endowed with one unit of time which they allocate between child-rearing and labor force participation. The preferences of members of generation t are defined over consumption as well as over the potential aggregate income of their children. Precisely, the utility of a generation t agent is given by

$$u_t = ln(c_t) + \beta ln(d_{t+1}) + \gamma ln(w_{t+1}h_{t+1}n_t) \tag{1}$$

where c_t and d_{t+1} are the adulthood and old age consumption respectively[6], n_t is the number of children chosen by individuals of generation t, h_{t+1} represents the level of human capital of the children, and w_{t+1} is the wage per efficiency unit of labor at time $t + 1$. The parameter $\beta > 0$ is the psychological discount factor and $\gamma > 0$ represents the altruism factor. The role of old-age consumption is to provide a motive for savings and therefore generate an endogenous supply of capital.

For a member of generation t, let $\phi \in (0, 1)$ and $\delta \in (0, 1)$ be the time cost of raising and educating a child respectively and ψ be the monetary cost of raising a child. While middle aged in period t (parenthood), individual faces the budget constraint:

$$c_t + s_t + \psi n_t + e_{t+1}n_t w_t \bar{h}_t = w_t h_t \left(1 - (\phi + \delta e_{t+1})\, n_t\right). \tag{2}$$

We have both time cost and monetary cost of educating and raising a child in order to determine the complete effect of income on fertility. A member of generation t is endowed with h_t unit of human capital at time t and w_t is wage rate per unit of human capital. Therefore, potential income of member of generation t is defined as $y_t \equiv w_t h_t$. Average human capital in the population is equal to teachers' average human capital and is represented by $\bar{h}_t$ and therefore, monetary cost of education per child is given by $e_{t+1}w_t\bar{h}_t$ where e_{t+1} is the schooling time per child. The budget constraint for the old-age is as follows:

$$d_{t+1} = R_{t+1}s_t. \tag{3}$$

An agent in generation t maximizes her utility as defined in (1) subject to period-wise budget constraints (2) and (3) by choosing saving s_t, number of children n_t and schooling time per child e_{t+1}.

3.3 Production of human capital

The level of human capital of children of members of generation t, h_{t+1}, is constructed in the following way so that it combines the features of Galor and Weil (2000), Galor

[6]The quantitative outcome will remain unchanged if we multiply the part of utility which represents the adulthood and old age consumption by $1 - \gamma$ as in Galor and Weil (2000).

and Moav (2002) and de la Croix and Doepke (2003):

$$h_{t+1} = \frac{B_t(\theta + e_{t+1})^{\eta(g_{t+1})}(h_t)^{\tau}(\bar{h}_t)^{\vartheta}}{1 + (g_{t+1})^{\xi}}. \tag{4}$$

In the above specification, h_{t+1} is an increasing function of their education e_{t+1} and a decreasing function of progress in the state of technology from period t to $t+1$, $g_{t+1} \equiv \frac{A_{t+1}-A_t}{A_t}$. The parameter $\tau \in [0,1]$ captures the inter generational transmission of human capital within the family, whereas $\vartheta \in [0, 1-\tau]$ represents externalities at the societal level. Crucial to our analysis, $\eta \geq 0$ captures the returns to education investment as mentioned above and, $\theta > 0$ ensures that children have some human capital even without parental investment in education. The efficiency parameter B_t is defined similar to de la Croix and Doepke (2003):

$$B_t = B(1+\rho)^{(1-\tau-\vartheta)t}.$$

We now define the relative human capital of household as $a_t = \frac{h_t}{\bar{h}_t}$. With this representation, h_{t+1} now depends on relative human capital a_t which parents are endowed with and, the average human capital of the society $\bar{h}_t$. Further, we are interested in endogenous growth, therefore, as in Rangazas (2000), equation (4) is compatible with endogenous growth for $\vartheta = 1 - \tau$, that is, finally

$$h_{t+1} = \frac{B(\theta + e_{t+1})^{\eta(g_{t+1})}(a_t)^{\tau}(\bar{h}_t)}{1 + (g_{t+1})^{\xi}}, \tag{5}$$

where $h(e_{t+1}, g_{t+1}, a_t, \bar{h}_t) > 0$, with $h_e > 0$, $h_{ee} < 0$, $h_g < 0$, $h_{gg} > 0$, and $h_{eg} > 0$. Our assumption that technological progress compliments rate of human capital due to an increase in education, that is $h_{eg} > 0$, implies that η increases with the level of technological progress. Thus, crucially, η in our setup is a function of technological progress, that is, $\eta(g_{t+1})$ with $\eta'(g_{t+1}) > 0$. Our assumption of $h_{eg} > 0$ will not be satisfied if η is assumed to be exogenous (constant), as for example, in de la Croix and Doepke (2003). Thus, individual human capital is an increasing and strictly concave function of education, and a decreasing and strictly convex function of the rate of technological progress. Moreover, technological progress increases the return to education in terms of human capital. We have empirically tested this assumption using Penn World Tables 10.1 and Barro and Lee (2018) dataset and find that there exists a significant and positive relationship between technological progress and returns to education.[7]

[7]See Appendix B.7 for empirical evidence supporting this assumed relationship.

3.4 Technological progress

The rate of technological progress g_{t+1} effectively depends on the education e_t and the population size P_t of the working generation in period t, precisely

$$g_{t+1} = \frac{A_{t+1} - A_t}{A_t} = g(e_t, P_t), \tag{6}$$

where for $e_t > 0$ and $P_t > 0$, $g(0, P_t) > 0$, along with $g_i(e_t, P_t) > 0$ and $g_{ii}(e_t, P_t) < 0$ for $i = e_t, P_t$. Hence, the rate of technological progress between time t and $t+1$ is a positive, increasing and strictly concave with respect to both the size and the level of education of the working generation at time t. Note that g is positive even if labor quality is zero. This assumption ensures a positive rate of technological progress during the Malthusian regime when people were investing nothing in quality at the extreme but the population size was sufficiently large.

3.5 Quantity-Quality trade off: Initial income or technology

As mentioned above, members of generation t chooses the number of their children, n_t, quality of their children, e_{t+1} and therefore their own consumption, so as to maximize their inter temporal utility function. Further, we assume that life-time consumption may be constrained by $\bar{c}$, precisely,

$$c_t + \frac{d_{t+1}}{R_{t+1}} \geq \bar{c}, \tag{7}$$

which, using (2) and (3), essentially gives us the following:

$$w_t \bar{h}_t \left(a_t(1 - (\phi + \delta e_{t+1})n_t) - e_{t+1} n_t \right) - \psi n_t \geq \bar{c}. \tag{8}$$

The optimization problem of a member of generation t is given by

$$\{n_t, e_{t+1}, s_t\} = argmax\{ln(c_t) + \beta ln(d_{t+1}) + \gamma ln(w_{t+1} h_{t+1} n_t)\}$$

subject to (8) along with

$$(n_t, e_{t+1}) \geq 0,$$

given (5) and (6).

It is easy to verify that the first order condition with respect to n_t implies that, as long as relative human capital a_t is sufficiently high so as to ensure that $c_t + s_t \geq \bar{c}$, the time spent by individual t raising a children is $\frac{\gamma}{(\psi + e_{t+1} w_t \bar{h}_t)}$. However, for individuals with low levels of a_t, the subsistence constraint (7) binds. They save a constant amount of income, consume the subsistence level $\bar{c}$, and use rest of the time for child rearing.

Precisely, the optimal amount of savings by members of generation t is given by,

$$s_t = \begin{cases} \frac{\beta\bar{c}}{(1+\beta)}, & \text{if } \frac{\bar{c}}{w_t\bar{h}_t} \le a_t < \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t}, \\ \frac{\beta w_t h_t}{(1+\beta+\gamma)}, & a_t \ge \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t}. \end{cases}$$

We now focus on two important decisions by members of generation t - optimal n_t and e_{t+1} - both of which depend on the different levels of initial $w_t\bar{h}_t$.[8] We provide the base case with the situation where initial level of potential income is very low (under a certain cutoff ω as defined below).[9]

3.5.1 Countries too poor to start with

We consider the case where countries are initially too poor. In our framework, this is represented by $w_t\bar{h}_t < \omega \equiv \frac{\phi\eta(g_{t+1})-\theta\delta}{\theta}\left[\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)} + \frac{\eta(g_{t+1})\psi}{\phi\eta(g_{t+1})-\theta\delta}\right]$, meaning the country's initial level of potential income is very low — specifically, less than ω. Under this scenario, the optimal level of fertility n_t with respect to a_t is as follows:

$$n_t = \begin{cases} \frac{(a_t w_t\bar{h}_t-\bar{c})}{(a_t\phi w_t\bar{h}_t+\psi)} & \text{if } \frac{\bar{c}}{w_t\bar{h}_t} \le a_t < \frac{\theta}{(\phi\eta(g_{t+1})-\theta\delta)} - \frac{\eta\psi}{(\phi\eta(g_{t+1})-\theta\delta)w_t\bar{h}_t}, \\ \frac{(a_t w_t\bar{h}_t-\bar{c})(1-\eta(g_{t+1}))}{(a_t\phi w_t\bar{h}_t+\psi-w_t\bar{h}_t\theta(1+\delta a_t))} & \text{if } \frac{\theta}{(\phi\eta(g_{t+1})-\theta\delta)} - \frac{\eta\psi}{(\phi\eta(g_{t+1})-\theta\delta)w_t\bar{h}_t} \le a_t < \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t}, \\ \frac{\gamma(1-\eta(g_{t+1}))a_t w_t\bar{h}_t}{(1+\beta+\gamma)\left[a_t\phi w_t\bar{h}_t+\psi-w_t\bar{h}_t\theta(1+\delta a_t)\right]} & \text{if } \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t} \le a_t. \end{cases} \tag{9}$$

Further, investment in education of their children e_{t+1} is given by

$$e_{t+1} = \begin{cases} 0, & \text{if } \frac{\bar{c}}{w_t\bar{h}_t} \le a_t < \frac{\theta}{(\phi\eta(g_{t+1})-\theta\delta)} - \frac{\eta\psi}{(\phi\eta(g_{t+1})-\theta\delta)w_t\bar{h}_t}, \\ \frac{\eta(g_{t+1})(a_t\phi w_t\bar{h}_t+\psi)-w_t\bar{h}_t\theta(1+\delta a_t)}{w_t\bar{h}_t(1-\eta(g_{t+1}))(1+\delta a_t)}, & \text{if } \frac{\theta}{(\phi\eta(g_{t+1})-\theta\delta)} - \frac{\eta\psi}{(\phi\eta(g_{t+1})-\theta\delta)w_t\bar{h}_t} \le a_t. \end{cases} \tag{10}$$

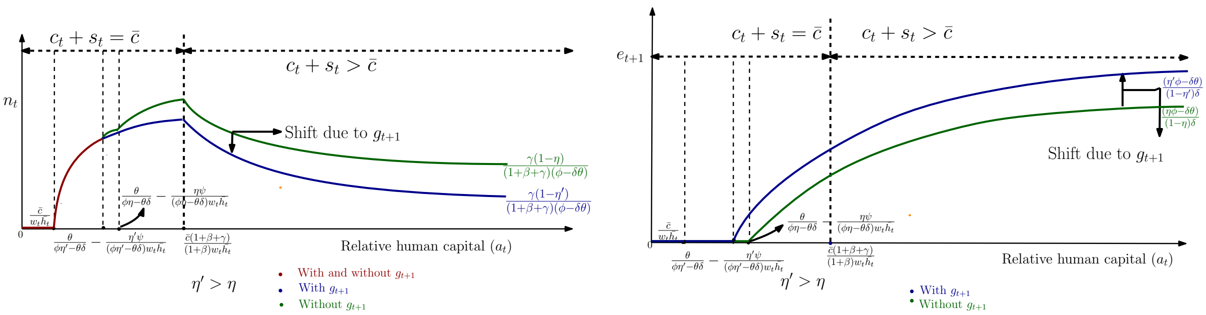
Potential income in period t is determined by the combination of fixed income, $w_t\bar{h}_t$, which remains constant across all households, and relative human capital, a_t, which varies from one household to another. The income-fertility trajectory of households in period $t+1$ is influenced by the dominance of either fixed income or relative human capital. Figure (3a) and (3b) below represent the paths of fertility and parental investment in education with respect to relative human capital. In Figure (4a) and (4b) we present the effect of fixed income on the paths of fertility and parental investment in education, keeping relative human capital fixed. With all these, finally Figure (5) shows the paths of fertility and parental investment in education with respect to po-

[8] We ignore the trivial case $0 \le a_t < \frac{\bar{c}}{w_t\bar{h}_t}$ for which the optimal choice of $\{n_t, e_{t+1}, s_t\}$= {0,0,0,0}.

[9] The knife-edge case and the case when countries are not very poor to start with that is, $w_t\bar{h}_t$ is greater than or equal to the cutoff has been presented in the **Appendix B.1**.

tential income.

With an increase in the level of fixed income, the optimal choice of fertility rises while education decreases, as the monetary cost of educating a child increases. Households with a lifetime subsistence consumption constraint, $c_t + s_t = \bar{c}$ - those countries operating under a Malthusian regime - will exhibit a positive income-fertility relationship. This implies that when the initial level of relative human capital satisfies $\frac{\bar{c}}{w_t \bar{h}_t} \leq a_t \leq \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t \bar{h}_t}$, an increase in fixed income and relative human capital will lead to a rise in the optimal fertility choice. After Malthusian regime, that is, given the initial level of relative human capital satisfies $\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t \bar{h}_t} \leq a_t$, with a rise in relative human capital the optimal choice of fertility decreases but the level of education increases, since the opportunity cost of raising children at the margin rises. During the **MG** regime - when $\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t \bar{h}_t} \leq a_t \leq (w_t \bar{h}_t)\left(\frac{\theta w_t \bar{h}_t - \psi}{\psi}\right)$ - a household's optimal choice of fertility decreases while education increases with a rise in potential income, as the net returns from raising more children are negative, whereas the returns from educating them are positive. Further, during the **MMG** regime - when $a_t \geq (w_t \bar{h}_t)\left(\frac{\theta w_t \bar{h}_t - \psi}{\psi}\right)$ - a household's optimal choice of fertility increases, while education initially rises and then falls with an increase in potential income. This occurs because the net returns from raising more children are positive, whereas the net returns from educating children are initially positive. However, at very high levels of potential income, the monetary cost of educating a child increases, causing the net returns from educating a child to become negative. Therefore, optimal choice of educational investment falls with rise in potential income. It might be the case that during **MMG** net returns from raising more kids is zero. Hence, during this regime income-fertility path may stay constant. Note that Cordoba and Ripoll (2014), Cordoba and Ripoll (2019) and Cordoba et.al. (2016) find that in a purely altruistic setup where no consumption constraint is imposed, the negative income-fertility relationship demands a restriction that the intergenerational elasticity of substitution to be greater than unity.

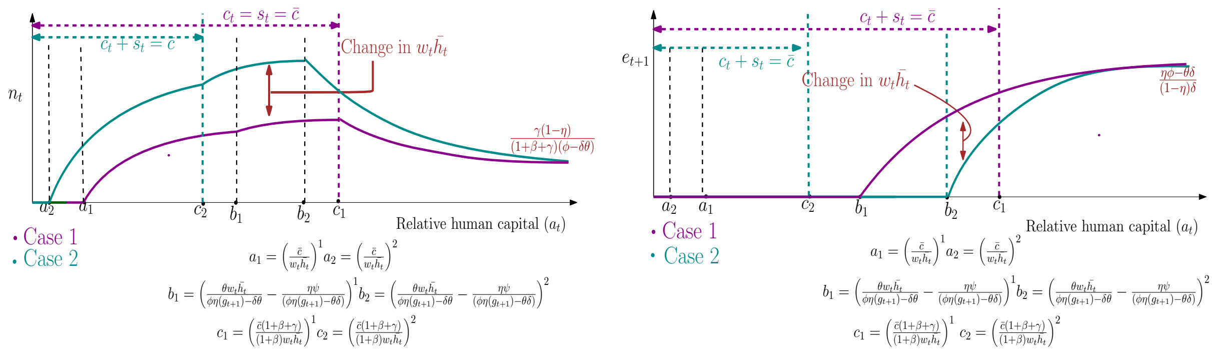


(a) Relative human capital - Fertility relationship

(b) Relative human capital - Education relationship

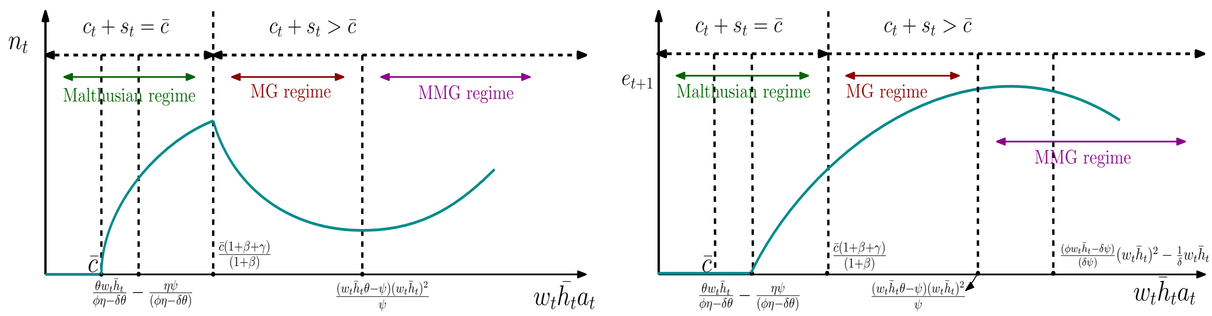
Figure 3: Effect of an increase in relative human capital, a_t.

According to Galor and Weil (2000) and Galor and Moav (2002), if the potential



(a) Relative human capital - fertility curve. (b) Relative human capital - education curve.

Figure 4: Effect of an increase in fixed income, $w_t\bar{h}_t$.



(a) Potential income - fertility curve. (b) Potential income - education curve.

Figure 5: Effect of an increase in potential income, $w_t\bar{h}_t a_t$.

income of parents exceeds $\tilde{z} = w_t h_t = \frac{\bar{c}}{1-\gamma}$, an increase in income neither raises the number of children nor improves their quality, as parents only incur the time cost of raising children. In their framework, time spent on raising children remains constant at γ, regardless of the increase in potential income. Consequently, the division of child-rearing time between quantity and quality remains unaffected. In contrast, we conclude that if potential income of parents exceeds $\tilde{z} = w_t h_t = \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)}$, an increase in potential income reduces the number of children while improving their quality when the opportunity cost of raising a child outweighs the monetary cost of educating them. At higher levels of potential income, an increase in income may lead to a rise in both the quantity and quality of children. The outcome depends on the relative strength of opportunity costs versus monetary costs, which in turn is determined by country-specific parameter values. The simultaneous increase in quality and quantity, observed at very high income levels, has been documented recently by Doepke et al. (2023). Over time, with further income growth, the number of children will rise while investment in education may decline, as the monetary cost of educating a child begins to dominate the opportunity cost of raising one. Thus, the **EP** outcome described by Jones (2022) may manifest, even under conditions of positive population growth.

In our analysis, η is positively linked to technological progress (g_{t+1}). With constant income, technological progress increases educational investment (e_{t+1}) and reduces fertility (n_t), as higher η raises the return on education. This shift begins early with technological advancements. Keeping relative human capital (a_t) fixed, an increase in g_{t+1} boosts e_{t+1} and lowers n_t, as shown in Figures (3a) and (3b). In Figure (3b), technological growth shifts the income-education curve upward and moves the threshold for positive education investment leftward. In Figure (3a), technological progress shifts the *income-fertility* curve downward. Additionally, the impact of g_{t+1} on e_{t+1} is stronger for households with higher a_t, evidenced by $\frac{de_{t+1}}{dg_{t+1}} > 0$ and $\frac{de_{t+1}}{da_t dg_{t+1}} > 0$. This intensifies the trade-off for those prioritizing quality over quantity.

Note that income plays a crucial role in determining whether households prioritize quantity or quality. Technological progress, on the other hand, influences the strength of this trade-off. Furthermore, with technological advancements, the *income-fertility* curve shifts downward, and the education-fertility curve shifts upward. This is because technological progress increases the incentive to invest in a child's education by enhancing the returns to education. With above results, we now present our proposition below.

Proposition 1. *Our framework can generate the income-fertility path, which initially ascends, then declines, and finally exhibits a modest uptick at very high income levels, aligning with empirical observations.*

Proof 1. *See Appendix **A.1**.*

We now explain how do we connect this model that generates empirically observed income-fertility path with the studies we have mentioned above. We confirm that we can generate the desired path at least from the two following modeling choices: (i) when we bring inequality and monetary cost of having children in Galor and Weil (2000) and, (ii) when we bring consumption constraint, time cost of educating and monetary cost of raising a child in de la Croix and Doepke (2003). Incorporating inequality through education costs, represented by $e_{t+1}w_t\bar{h}_t$, highlights that education is relatively more expensive for poorer parents. Galor and Weil (2000) predicts a declining *income-fertility* curve for skilled parents because, while education costs are fixed, the time or opportunity cost of raising additional children increases with income. Consequently, skilled parents with high relative human capital experience a declining relationship between human capital and fertility. On the other hand, the model by de la Croix and Doepke (2003) does not address how inequality impacts fertility during the Malthusian regime. By integrating factors like consumption constraints, the monetary cost of raising a child, and the time cost of education into their framework, we observe a positive effect of higher relative human capital on fertility during the Malthusian period. Furthermore, these additions enable an analysis of how monetary and opportunity costs influence the trade-off between having more children and investing in their quality. Further, as mentioned earlier, unlike these studies, our findings are also compatible with the recent observation made by Doepke et al. (2023) that the *income-fertility* relationship has recently shown a slight upward trend. In our model, the emerging trend is driven by inequality dynamics and technological growth, captured through the fixed monetary cost of education, with or without childcare services. Childcare is closely linked to inequality, as only the wealthy, who face relatively higher education costs can afford it and benefit by converting opportunity costs into monetary costs. This weakens the impact of opportunity cost, and fertility starts rising as fixed income gains outweigh opportunity cost effects. Therefore any such services can well be explained through inequality dynamics only.

4 Inequality affecting fertility, education and growth

We now explicitly incorporate inequality into the above analysis by considering the fact that each household is endowed with a different level of relative human capital, a_t. This aims to determine the role of inequality in explaining the transition from the Malthusian regime to the **MG** regime, and subsequently to the **MMG** regime. With the evolution of education, e_t, and technological progress, $g_{t+1}(e_t, P_t)$, inequality - defined as an increase in the spread of income distribution - also increases. From (5),

$$a_{t+1} = \frac{h_{t+1}}{\bar{h}_{t+1}} = \frac{B(\theta + e_{t+1})^{\eta(g_{t+1})} a_t^{\tau} \bar{h}_t}{(1 + g_{t+1}^{\xi})\bar{h}_{t+1}}.$$

For a given dispersion of relative human capital, a_t, an increase in parents' education, e_t, leads to a rise in technological progress during their children's generation, denoted by $g_{t+1}(e_t, P_t)$. This implies that parents' investment in their children's education, e_{t+1}, also increases. With the rise in technological progress, the investment in children's education in relation to relative human capital remains positive, i.e., $\frac{de_{t+1}}{da_t dg_{t+1}} > 0$. Consequently, a_{t+1} increases at different rates for different households, depending on the initial level of a_t.

We observe that, with the evolution of education, e_t, $\bar{h}_t$ remains unchanged, while the average human capital in period $t + 1$, $\bar{h}_{t+1}$, changes in a similar magnitude for all households. However, as parents' education evolves, the relative human capital of children with wealthier parents rises more significantly because their initial level of relative human capital, a_t, is higher. Therefore, with technological progress, the positive effect of increased education is much greater for wealthy households. The overall effect of technological progress leads to a rise in a_{t+1}. In contrast, for poorer households, the positive effect of increased education is much smaller, and the overall effect of technological progress may result in a slight rise or fall in a_{t+1}. Hence, inequality increases as the disparity in relative human capital widens. Thus, with the evolution of e_t, both technological progress, $g_{t+1}(e_t, P)$, and inequality (e_t) increase for a given population level. Specifically, human capital is assumed to be distributed among the adult population according to the uniform distribution function $F_t(h_t)$. The total population in period t, P_t, evolves over time according to

$$P_{t+1} = P_t \int_0^\infty n_t \, dF_t(h_t). \tag{11}$$

Average human capital $\bar{h}_t$ is given by

$$\bar{h}_t = \int_0^\infty h_t \, dF_t(h_t). \tag{12}$$

Market clearing conditions for capital and labor are

$$K_{t+1} = P_t \int_0^\infty s_t \, dF_t(h_t), \tag{13}$$

and

$$L_t = P_t \left[\int_0^\infty h_t(1 - (\phi + \delta e_{t+1})n_t) \, dF_t(h_t) - \int_0^\infty e_{t+1} n_t \bar{h}_t \, dF_t(h_t) \right]. \tag{14}$$

Equation (14) reflects the fact that time available for teaching is not available for goods production. If we assume each household is endowed with same level of human capital, that is there is no inequality, $a_t = a$ then Equation (11)-(14) can be written as

$$P_{t+1} = P_t n_t,$$

that is, $n_t - 1$ is the rate of population growth. Further,

$$\bar{h}_t a = h_t,$$

$$K_{t+1} = P_t s_t,$$

and

$$L_t = P_t h_t \left[1 - (\phi + \delta e_{t+1})n_t - (e_{t+1} n_t)/a\right].$$

The state of the technology at time $t + 1$, A_{t+1}, is defined as,

$$A_{t+1} = (1 + g_{t+1})A_t.$$

The distribution function of human capital, $F_t(h)$ evolves according to,

$$F_{t+1}(h) = \frac{P_t}{P_{t+1}} \int_0^\infty n_t \mathbb{I}(h_{t+1} \leq h)\, dF_t(h_t), \tag{15}$$

where $\mathbb{I}(.)$ is an indicator function. If households are endowed with different level of relative human capital, that is there is an inequality in the economy, then relative human capital is defined as $a_t = \frac{h_t}{\bar{h}_t}$. We define the distribution of the relative human capital level as,

$$G_t(a_t) \equiv F_t(a_t, \bar{h}_t).$$

Therefore, equation (11), (12) and (15) can be written as,

$$N_t = \int_0^\infty n_t\, dG_t(a_t),$$

$$G_{t+1}(a) = \frac{1}{N_t} \int_0^\infty n_t \mathbb{I}(a_{t+1} \leq a)\, dG_t(a_t),$$

and

$$1 = \int_0^\infty a_t\, dG_t(a_t).$$

Crucially, in our setup human capital is distributed according to the uniform distribution function $F_t(h_t)$. If h_t is assumed to have a support $[\underline{h_t}, \bar{h}_t]$, we can measure differential fertility as $\frac{n_t^{\text{poor}(\underline{h_t})}}{n_t^{\text{rich}(\bar{h}_t)}}$.[10] de la Croix and Doepke (2003) link inequality to growth via differential fertility, total fertility, and weighted average education. They conclude that a mean-preserving spread in income distribution increases fertility differentials, reducing weighted average education, and thus growth. Our setup explains how the initial distribution of human capital connects inequality to growth through these channels.

With the evolution of education and technological progress, inequality rises as the spread of the distribution increases. Assuming an economy lies within Malthusian regime and households are optimally choosing positive level of investment in child's education. Due to the shape of the fertility and education curve, rising inequality leads to an increase in differential fertility, total fertility, and average education.[11]

[10]in calibration, following de la Croix and Doepke (2003), differential fertility is measured by the difference in top and bottom income quintiles.

[11]For instance, if two types of parents initially invest n_1 and n_2 in fertility and e_1 and e_2 in education, where $n_2 > n_1$ and $e_2 > e_1$, total fertility is $n_1 + n_2$, total education is $e_1 + e_2$, and weighted average education is $\frac{n_1 e_1}{n_1+n_2} + \frac{n_2 e_2}{n_1+n_2}$. As inequality grows, n_1 and e_1 increase to n_1' and e_1', while n_2 and e_2 increase more substantially to n_2' and e_2', resulting in higher total fertility, total education, and weighted average education. Notably, the disproportionate rise in n_2 and e_2 compared to n_1 and e_1 implies $\frac{n_2'}{n_1'+n_2'} > \frac{n_2}{n_1+n_2}$, driving the increase in weighted average education. Thus, rising inequality during this regime boosts differential fertility, total fertility, and weighted average education.

In the **MG** regime, where fertility declines with rising potential income, increasing inequality raises differential fertility while reducing total fertility and weighted average education.[12] In the **MMG** regime, an increase in inequality raises differential fertility, total fertility, and weighted average education. With a mean-preserving spread, fertility increases more for those who initially invest more in education, resulting in a rise in weighted average education (as $\frac{n_2'}{n_2} > \frac{n_1'}{n_1}$ following the notational expressions in the above two footnotes). Lower fertility among individuals with less human capital during the Malthusian and **MMG** regimes amplifies inequality's positive effect on human capital accumulation. However, as education evolves further, investment in children's education begins to decline with potential income. Consequently, increasing inequality causes total fertility to rise but weighted average education to fall. While de la Croix and Doepke (2003) focus on inequality's effect on growth via differential fertility during the **MG** regime, incorporating the Malthusian and **MMG** regimes suggests that rising inequality may actually enhance growth. Thus, we are in a position to present our next proposition as follows.

Proposition 2. *As inequality increases, the following observations hold true:*

- *in the Malthusian regime, total fertility and weighted average education increase,*
- *in the **MG** regime, total fertility and weighted average education decline, and*
- *in the **MMG** regime, total fertility and weighted average education increase, resembling the Malthusian regime.*

Proof 2. *The proof follows directly from above.*

There is no consistent negative relationship between differential fertility and growth during the Malthusian and **MMG** regimes. Economies can move from the MG regime to the more prosperous **MMG** regime as fertility begins to rise with potential income. Interestingly, economies may also regress to the Malthusian regime under certain conditions, leading history to repeat itself. As we show, importantly, inequality plays a key role in preventing such regression. These dynamics, including the resemblance of the possibility outlined in Jones (2022) within our framework, are detailed in the next section.

[12]Consider two types of parents specified exactly as in the previous footnote. After an increase in inequality, n_1 and n_2 fall to n_1' and n_2', reducing total fertility to $(n_1' + n_2')$, while e_1 and e_2 rise to e_1' and e_2', increasing total education to $(e_1' + e_2')$. As inequality grows, the fall in n_2 and rise in e_2 outweigh the changes in n_1 and e_1, leading to $\frac{n_2'}{n_1'+n_2'} < \frac{n_2}{n_1+n_2}$ and $\frac{n_1'}{n_1'+n_2'} > \frac{n_1}{n_1+n_2}$, i.e., $\frac{n_2'}{n_2} < \frac{n_1'}{n_1}$. Consequently, weighted average education declines with rising inequality. Thus, in the **MG** regime, inequality increases differential fertility while reducing total fertility and weighted average education.

5 Empirical support of the basic mechanism

We support our mechanism with a simple empirical observation: historical fertility patterns are shaped by technological progress through its impact on education and finally inequality. As technological progress increases, the returns to education rises, leading to higher educational expenditure and lower fertility. Since wealthy families can invest more in education than poorer families, the relative human capital of their children grows disproportionately, thereby widening inequality. The effect of rise in inequality on fertility and educational investment, however, depends on the specific regime in which an economy operates.

5.1 Empirical method and Data

5.1.1 Model specification

We estimate the model using a two-stage least square (2SLS) approach to examine the impact of inequality on fertility and educational expenditure through technological progress. In the first stage, we estimate how technological progress influences inequality, using total factor productivity (TFP), which is a measure of technological progress as an instrument for inequality,

$$log(\text{Inequality}_{i,t}) = \lambda_1 + \lambda_2 \text{TFP}_{i,t} + \lambda_3 log(\text{X}_{i,t}) + \epsilon_{i,t}. \tag{16}$$

where $\text{X}_{i,t}$ denotes the set of control variables, which includes the share of black population and the share of urban population. In the second stage, we examine how inequality influences fertility and educational expenditure by estimating the following equation:

$$log(\text{Fertility}_{i,t}) = \beta_1 + \beta_2 log(\widehat{\text{Inequality}}_{i,t}) + \beta_3 log(\text{X}_{i,t}) + u_{i,t}. \tag{17}$$

$$log(\text{Education}_{i,t}) = \delta_1 + \delta_2 log(\widehat{\text{Inequality}}_{i,t}) + \delta_3 log(\text{X}_{i,t}) + v_{i,t}. \tag{18}$$

where i denotes the state and t denotes time. Technological progress is measured using total factor productivity (TFP), which is calculated following Turner et al. (2013). Equation (16) presents the first-stage regression, which identifies how TFP of state i in period t influences inequality. Equations (17) and (18) present the second-stage regressions, which examine how inequality affects fertility and educational expenditure, respectively. To check how technological progress impacts fertility either directly or through inequality and kid's educational investment we estimate the following equation using an OLS approach:

$$log(\text{Fertility}_{i,t}) = \gamma_1 + \gamma_2 log(\text{Inequality}_{i,t}) + \gamma_3 \text{TFP}_{i,t} + \gamma_4 log(\text{Education}_{i,t}) + \gamma_5 log(\text{X}_{i,t}) + z_{i,t}. \quad (19)$$

The coefficient, γ_3, measures the direct effect of total factor productivity on fertility.

5.1.2 Data

We have used the data constructed by Galor et al. (2009) and Turner et al. (2013). The dataset cover 28 states of USA for four period 1880, 1900, 1920 and 1940. Technological progress is measured using total factor productivity (TFP) where it has been calculated using output per worker, physical capital per worker and human capital per worker using Turner et al. (2013). The other variables are calculated using Galor et al. (2009). Inequality is measured as land share of large farms. They track how the share of land held by the largest number of farms evolve over time, where the number of these farms is held constant at farms held in 1880. This means that the share has been calculated as $S_{i,t} = 1 - (1 - \text{Topfarms}_{i,1880}/\text{Farms}_{i,t})^{\varrho_{i,t}}$ where $\text{Farm}_{i,t}$ measures the total number of farms in year t, and $\varrho_{i,t}$ measures the coefficient on the Lorenz curve from year t in state i. Education expenditure is obtained from the Historical Statistics of the U.S. and from the U.S. Bureau of Education, Report of the Commissioner of Education. The general fertility rate is calculated as the total number of live births per 1,000 females of reproductive age between 15 and 44 years in a population per year. General fertility rate, percentage of black population and urban population is taken from the U.S. census.

5.2 Estimation results

We use 2SLS approach to show how our mechanism works. This is because the model satisfies the criteria as indicated by the F-statistics (column (1)-(4)) of Wu-Hausman test for endogeneity in both the Table 1 and 2. Hence, OLS estimates would be inconsistent, and therefore we use the IV approach for reliable results. This approach proceeds in two steps. In the first step, the quality of instrument is assessed. We have taken TFP as an instrument for inequality. In the next step we see how inequality affects fertility and educational expenditure. The 2SLS regression are presented in Table 1 and 2 where the second stage regression results are presented in upper panel and first stage regression results are presented in lower panel. Table 1 shows how well TFP estimates inequality and how inequality affects fertility. Table 2 shows how well TFP estimates inequality and how inequality affects educational expenditure. Table 3 shows the main part of the mechanism which says that technological progress is the channel through which school attainment and inequality affects fertility. It will not affect fertility directly. First

Table 1: IV Regressions.

	1	2	3	4
		Second Stage (Fertility)		
blackpopulation		1.165***		0.535*
		[0.000]		[0.026]
urbanpopulation			-1.17***	-0.98***
			[0.000]	[0.000]
Inequality	3.91**	3.85**	3.14**	3.15**
	[0.006]	[0.003]	[0.004]	[0.004]
Instruments		**First Stage (Inequality)**		
TFP	0.000009***	0.000009***	0.00001***	0.00001***
	[0.000]	[0.000]	[0.000]	[0.000]
		Diagnostic Tests		
Weak Instruments	26.57***	28.93***	32.12***	32.21***
	[0.000]	[0.000]	[0.000]	[0.000]
Wu-Hausman	58.54***	56.55***	42.31***	42.05***
	[0.000]	[0.000]	[0.000]	[0.000]
Observations	175	172	167	167

p-values in parentheses.
*** $p<0.001$, ** $p<0.01$, * $p<0.05$.

Table 2: IV Regressions.

	1	2	3	4
		Second Stage (Education)		
blackpopulation		-6.17***		-3.81***
		[0.000]		[0.000]
urbanpopulation			5.08***	3.74***
			[0.000]	[0.000]
Inequality	-19.56***	-18.84***	-17.49***	-17.56***
	[0.000]	[0.000]	[0.000]	[0.000]
Instruments		**First Stage (Inequality)**		
TFP	0.000009***	0.000009***	0.00001***	0.00001***
	[0.000]	[0.000]	[0.000]	[0.000]
		Diagnostic Tests		
Weak Instruments	25.13***	27.28***	32.23***	32.32***
	[0.000]	[0.000]	[0.000]	[0.000]
Wu-Hausman	85.35***	140.86***	98.06***	167.92***
	[0.000]	[0.000]	[0.000]	[0.000]
Observations	169	164	168	168

p-values in parentheses.
*** $p<0.001$, ** $p<0.01$, * $p<0.05$.

Table 3: Fertility Regression.

	1	2	3	4
blackpopulation		-0.56***		-0.60***
		[0.000]		[0.000]
urbanpopulation			0.08	0.145
			[0.494]	[0.22]
Inequality	-1.716***	-1.789***	-1.763***	-1.887***
	[0.000]	[0.000]	[0.000]	[0.000]
TFP	0.00001	-0.0000007	0.00001	-0.000004
	[0.119]	[0.9]	[0.208]	[0.60]
Education	-0.2***	-0.28***	-0.21***	-0.30***
	[0.000]	[0.000]	[0.000]	[0.000]
R Squared	0.57	0.60	0.57	0.60
Observations	169	164	168	168

p-values in parentheses.
*** $p<0.001$, ** $p<0.01$, * $p<0.05$.

consider the first stage regressions. We can see column (1)-(2) and column (3)-(4) in both Table 1 and 2 shows that 1 unit rise in TFP leads to 0.009% and 0.01% increase in fertility respectively and it is statistically significant at 0.1%. TFP satisfies the relevance criteria as indicated by F-statistics for weak instrument test (column(1)-(4)) in both the tables.

The second stage regression in the upper panel shows how inequality affect fertility and education expenditure. Table 1 shows that 1% percentage increase in inequality approximately leads to 3%-4% increase in fertility (column(1)-(4)) with as well as without any control variable. Table 2 shows that 1% percentage increase in inequality approximately leads to 17%-19% decrease in education expenditure (column(1)-(4)) with as well as without any control variable.[13] Table 3 shows whether TFP affects fertility directly or not. We found that the coefficient of TFP is insignificant once inequality and educational expenditure has been included in the regression analysis. Therefore, we can claim that the historical fertility path is determined by the two effects of technological progress: (1) effect on inequality (2) effect on education expenditure. The dominance of two effect determines whether fertility will rise or fall overtime.[14]

[13]We use data constructed by Madsen and Strulik (2023) covering the period 1700–2018. Technological progress is proxied by patent intensity, while inequality is measured as the gap between rental and wage income. Our results show that technological progress leads to a decline in fertility in both regimes. However, the effects of inequality on fertility and educational expenditure depend on the regime in which an economy operates. During the Malthusian regime, rising inequality increases both fertility and educational expenditure, whereas in the **Modern Growth (MG)** regime, both decline with increasing inequality.

[14]As a robustness exercise, we also constructed a standardized measure of total factor productivity (TFP) by subtracting the mean and dividing by the standard deviation. We find that the coefficient associated with standardized TFP in the first-stage regression increases substantially while remaining highly significant. All other results remain qualitatively unchanged, with differences only in the magnitude of

While the IV estimates in Table 1 show that higher inequality increases fertility, consistent with our mechanism, the OLS estimates in Table 3 turn negative once educational investment is controlled for. This pattern is entirely consistent with our theory: inequality raises fertility indirectly through its depressing effect on education. Once that channel is isolated, the residual direct effect of inequality is negative. Hence, the sign reversal reflects the presence of an education-mediated mechanism, not a contradiction. Thus, in a nutshell, while our main focus is not necessarily causal, IV estimates using TFP - plausibly exogenous and operating primarily through the allocation of resources across households - reveal a striking pattern: higher inequality substantially increases fertility and sharply reduces education. These effects are robust across specifications and supported by strong first-stage instruments, lending credibility to this interpretation under the exclusion restriction. Together, the results provide compelling empirical support for our theory, offering a powerful explanation of the observed historical *income-fertility* relationship.

6 Global dynamics

The development of an economy is shaped by various factors, including changes in output per worker, population growth, technological progress, education per worker, relative human capital, capital per effective worker, and levels of inequality. To isolate and examine the effect of technological progress on investment in kid's education alone, we first assume a scenario without inequality. In this context, all households are considered to have a constant level of relative human capital, denoted by $a_t = a$. As a result, the potential income of each household during period t can be expressed as $w_t \bar{h}_t a = w_t h_t$.

6.1 Dynamical system without inequality

Fixed income is expressed as $z_t(\kappa_t, e_t, g_t) = w_t \bar{h}_t = w_t(\kappa_t)\bar{h}_t(e_t, g_t)$, while technological progress is defined by $g_{t+1} = g(e_t, P_t)$. The dynamics of capital per effective worker, κ_t, is determined by the following equation:

the estimated coefficients.

$$\kappa_{t+1} = \begin{cases} A\left[\dfrac{a_t^{1-\tau}(1+g(e_t,P_t)^{\xi})\left(\bar{c}\phi+\psi\right)}{\theta^{\eta(g(e_t,P_t))}(1+g(e_t,P_t))(a_t z_t-\bar{c})}\right]\kappa_t \\ \qquad \equiv \Xi^a(\kappa_t,e_t,g_t,P_t)\kappa_t \text{ if } c_t+s_t=\bar{c}, \\ A'\left[\dfrac{(1+g(e_t,P_t)^{\xi})(a_t z_t\psi(1-\eta(g(e_t,P_t)))+\bar{c}(a_t\phi z_t+\eta(g(e_t,P_t))\psi-z_t\theta(1+\delta a_t)))}{a_t^{\tau}(1-\eta(g(e_t,P_t)))(1+g(e_t,P_t))(\theta+e_{t+1}(z_t,g_{t+1}))^{\eta(g(e_t,P_t))}z_t(a_t z_t-\bar{c})}\right]\kappa_t, \\ \qquad \equiv \Xi^b(\kappa_t,e_t,g_t,P_t)\kappa_t \text{ if } c_t+s_t=\bar{c}, e_{t+1}>0 \\ B\left[\dfrac{(1+g^{\xi}(e_t,P_t))\left[(z_t(.)(\phi a_t-\theta(1+\delta a_t))(1+\beta)+(1+\beta+\gamma(1-\eta(g(e_t,P_t))))\psi)\right]}{a_t^{\tau}(1-\eta(g(e_t,P_t)))(1+g(e_t,P_t))(\theta+e_{t+1}(z_t,g(e_t,P_t)))^{\eta(g(e_t,P_t))}}\right]\kappa_t \\ \qquad \equiv \Xi^c(\kappa_t,e_t,g_t,P_t)\kappa_t \text{ if } c_t+s_t>\bar{c}, e_{t+1}>0, \end{cases} \tag{20}$$

where as follows from the fact that fixed income rises as capital per effective worker rises, that is, $\frac{dz_t}{d\kappa_t}>0$, $\Xi_{\kappa}^{\{a,b\}}(e_t,g_t,\kappa_t,P_t)<0$, $\Xi_{\kappa}^{\{c\}}(e_t,g_t,\kappa_t,P_t)>0$, $\Xi_e^{\{a\}}(e_t,g_t,\kappa_t,P_t)=0$, $\Xi_e^{\{b\}}(e_t,g_t,\kappa_t,P_t)<0$, $\Xi_e^{\{c\}}(e_t,g_t,\kappa_t,P_t)>0$ if $a\le\bar{a}$ and $\Xi_e^{\{c\}}(e_t,g_t,\kappa_t,P_t)<0$ if $a\ge\bar{a}$. The dynamics of education level, e_t, is given by,

$$e_{t+1} = \begin{cases} e_t=0 \quad \text{if } c_t+s_t=\bar{c}, \\ \dfrac{\eta(g(e_t,P_t))(a_t z_t(.)\phi+\psi)-z_t(.)\theta(1+\delta a_t)}{z_t(.)(1-\eta(g(e_t,P_t)))}+e_t \\ -\dfrac{\eta(g(e_{t-1},P_{t-1}))(z_{t-1}(.)\phi+\psi)-z_{t-1}(.)\theta(1+\delta a_{t-1})}{z_{t-1}(.)(1-\eta(g(e_{t-1},P_{t-1})))} \equiv \Phi^a(\kappa_t,e_t,g_t,P_t)+e_t \text{ if } c_t+s_t=\bar{c}, \\ \dfrac{\eta(g(e_t,P_t))(a_t z_t(.)\phi+\psi)-z_t(.)\theta(1+\delta a_t)}{z_t(.)(1-\eta(g(e_t,P_t)))}+e_t \\ -\dfrac{\eta(g(e_{t-1},P_{t-1}))(a_{t-1}z_{t-1}(.)\phi+\psi)-z_{t-1}(.)\theta(1+\delta a_{t-1})}{z_{t-1}(.)(1-\eta(g(e_{t-1},P_{t-1})))} \equiv \Phi^b(\kappa_t,e_t,g_t,P_t)+e_t \text{ if } c_t+s_t>\bar{c}, \end{cases} \tag{21}$$

where as follows from the fact, $\frac{dz_t}{d\kappa_t}>0$, $\Phi_{\kappa}^{\{a,b\}}(e_t,g_t,\kappa_t,P_t)<0$, $\Phi_e^a(e_t,g_t,\kappa_t,P_t)>0$, $\Phi_e^b(e_t,g_t,\kappa_t,P_t)>0$ if $a_t\le\tilde{a}$ and $\Phi_e^b(e_t,g_t,\kappa_t,P_t)<0$ if $a_t\ge\tilde{a}$.

The dynamical system is categorized into four distinct regimes. In the first regime, the lifetime subsistence constraint binds and households optimally choose zero investment in education. The second regime retains this binding constraint and households choose to make positive investments in education. Both of these fall under the Malthusian regime. In the third regime, the constraint ceases to bind and fertility declines as potential income rises - this phase is referred to as the **MG** regime. Finally, in the fourth regime, the constraint remains non-binding, but fertility increases as potential income rises - this is known as the **MMG** regime. For a given population size (P), the economy's evolution across these four regimes is governed by a three-dimensional, non-linear, first-order autonomous system.

$$\begin{cases} \kappa_{t+1} = \Xi^i(e_t, g_t, \kappa_t; P)\kappa_t \\ e_{t+1} = \Phi^i(g(e_t, P), \kappa_t; P) + e_t \\ g_{t+1} = g(e_t; P), \end{cases}$$

where $e_t = 0$ and $\Phi^i(e_t, g_t, \kappa_t; P) = 0$ for the households in the first regime.

We will now analyze the economy's evolution from the Malthusian to the **MG** and then finally to the **MMG** regime. The phase diagrams include three key elements: the Conditional Malthusian Frontier, which separates regions where lifetime subsistence consumption is binding from those where it is not; the XX locus, representing pairs (e_t, κ_t) for which capital per effective labor is constant; and the EE locus, representing pairs where education per worker is constant. *The Malthusian Frontier* consists of triplets $(e_t \geq 0, g_t, \kappa_t)$ where individual income equals $\frac{\bar{c}(1+\beta+\gamma)}{1+\beta}$. *The Conditional Malthusian Frontier* be the set of all pairs of (e_t, κ_t) for which conditional on a given technological level, g_t, individual income is equal to $\frac{\bar{c}(1+\beta+\gamma)}{1+\beta}$,

$$MM|g_t \equiv \left((e_t, \kappa_t) : w_t h_t = (1-\alpha)\kappa_t^{\alpha} h_t(e_t, \bar{g}) = \frac{\bar{c}(1+\beta+\gamma)}{1+\beta} | g_t = \bar{g} \right).$$

If $(\kappa_t, e_t) \in MM|g_t$ then κ_t is a decreasing strictly convex function of e_t.

The XX Locus is the locus of all triplets (e_t, κ_t, g_t) such that for a given population size, the capital per effective labor is in steady state, that is, from (20), $\Xi^i(e_t, g_t, \kappa_t; P) = 1$ $\forall\, i$. For the households in first regime, the capital per effective labor is in steady state if,

$$A \left[\frac{a_t^{1-\tau}(1+g(0,P_t)^{\xi})\left(\bar{c}\phi + \psi\right)}{\theta^{\eta(g(0,P_t))}(1+g_{t+1})(a_t z_t - \bar{c})} \right] = 1,$$

which means $\kappa_t = z^{-1}(\zeta)$ where ζ is some constant. This implies $\frac{d\kappa_t}{de_t} = 0$, hence for households with $c_t + s_t = \bar{c}$ and $e_t = 0$,

$$\kappa_{t+1} - \kappa_t \begin{cases} < 0, & \text{if } \kappa_t > z^{-1}(\zeta) \\ = 0, & \text{if } \kappa_t = z^{-1}(\zeta) \\ > 0, & \text{if } \kappa_t < z^{-1}(\zeta). \end{cases}$$

For households in second and third regime, the capital per effective labor is in steady state, if,

$$\Xi^b(e_t, g_t, \kappa_t; P) = 1 \quad \text{and} \quad \Xi^c(e_t, g_t, \kappa_t; P) = 1.$$

Since $\Xi^b(e_t, g_t, \kappa_t; P)$ is strictly monotonically decreasing and $\Xi^c(e_t, g_t, \kappa_t; P)$ is strictly monotonically increasing in κ_t, there exists a single valued function $\kappa_t = \kappa(e_t)$ such that $\Xi^{b,c}(e_t, g_t, \kappa_t; P) = 1$ and therefore, $(e_t, \kappa(e_t)) \in XX$. Now, $\Xi_e^{b,c}(e_t, g_t, \kappa_t; P)$ is not necessarily monotonic; $\kappa'(e_t)$ is not necessarily monotonic, hence for households with $c_t + s_t = \bar{c}, e_t > 0$,

$$\kappa_{t+1} = \kappa_t \begin{cases} = 0, & \text{if } \kappa_t = \kappa(e_t) \\ < 0, & \text{if } \kappa_t > \kappa(e_t) \\ > 0, & \text{if } \kappa_t < \kappa(e_t). \end{cases}$$

For households with $c_t + s_t > \bar{c}$,

$$\kappa_{t+1} - \kappa_t \begin{cases} = 0, & \text{if } \kappa_t = \kappa(e_t) \\ > 0, & \text{if } \kappa_t > \kappa(e_t) \\ < 0, & \text{if } \kappa_t < \kappa(e_t), \end{cases}$$

that is, the sign of $\frac{d\kappa_t}{de_t}$ depends on $\Xi_e^{\{i\}}(e_t, \kappa_t, g_t; P)$. Without loss of generality, in the space (e_t, κ_t) the locus XX for households in second and third regime is downward sloping and is upward sloping in the fourth regime.

The EE locus represents the triplets (e_t, g_t, κ_t) such that the quality of labor, e_t, is in steady state, meaning $e_{t+1} = e_t$. For households in the first regime, the EE locus is defined by:

$$e_{t+1} - e_t = 0,$$

which corresponds to a vertical line where $e_{t+1} = e_t = 0$. For households in second and third regime, using (21), the quality of labor e_t is in steady state if,

$$\Phi^a(e_t, g_t, \kappa_t; P) = 0,$$

where $\Phi_\kappa^a(e_t, g_t, \kappa_t; P) < 0$ and $\Phi_e^a(e_t, g_t, \kappa_t; P) > 0$. This implies $\frac{d\kappa_t}{de_t} = -\frac{\Phi_e^a(e_t, g_t, \kappa_t; P)}{\Phi_\kappa^a(e_t, g_t, \kappa_t; P)} > 0$. The sign of $\kappa'(e_t)$ is determined by two effects:

- The **substitution effect** $(\eta_g g_e)$, which is positive. As e_t increases, technological progress (g_{t+1}) rises, boosting η, which in turn increases e_{t+1}, w_{t+1}, and κ_{t+1}.
- The **income effect** (z_e), which is negative. An increase in e_t raises fixed income (z_t), leading to higher monetary costs of education and causing e_{t+1} to decline.

In these regimes, the substitution effect dominates the income effect, resulting in

$\kappa'(e_t) > 0$. Thus, the EE locus is an upward-sloping curve. Specifically:

$$e_{t+1} - e_t \begin{cases} = 0 & \text{if } e_t = e(\kappa_t) \\ > 0 & \text{if } e_t > e(\kappa_t) \\ < 0 & \text{if } e_t < e(\kappa_t). \end{cases}$$

In fourth regime, using (21), the quality of labor is in steady state, if, $\Phi^c(e_t, g_t, \kappa_t; P) = 0$. Here, $\Phi^c_\kappa(e_t, g_t, \kappa_t; P) < 0$, decreasing monotonically with κ_t. $\Phi^b_e(e_t, g_t, \kappa_t; P) > 0$, increases monotonically with e_t for $a_t < \hat{a}$ but decreases for $a_t > \hat{a}$, that is,

$$\frac{d\kappa_t}{de_t} = -\frac{\Phi^a_e(e_t, g_t, \kappa_t; P)}{\Phi^a_\kappa(e_t, g_t, \kappa_t; P)} \begin{cases} > 0 & \text{if } a_t < \hat{a} \\ < 0 & \text{if } a_t > \hat{a}. \end{cases}$$

In the fourth regime, the income effect dominates when $a_t > \hat{a}$, leading the EE locus to slope downward. Conversely, the locus slopes upward when $a_t < \hat{a}$. Hence,
if, $a_t < \hat{a}$,

$$e_{t+1} - e_t \begin{cases} = 0 & \text{if } e_t = e(\kappa_t) \\ > 0 & \text{if } e_t > e(\kappa_t) \\ < 0 & \text{if } e_t < e(\kappa_t), \end{cases}$$

and if, $a_t > \hat{a}$,

$$e_{t+1} - e_t \begin{cases} = 0 & \text{if } e_t = e(\kappa_t) \\ < 0 & \text{if } e_t > e(\kappa_t) \\ > 0 & \text{if } e_t < e(\kappa_t). \end{cases}$$

The dynamical system as depicted in Figure 6 in the space (e_t, x_t) is characterized by unique and globally stable steady state, referred to as the Conditional Malthusian steady state, occurring at $(e_t, \kappa_t) = (0, z^{-1}(\zeta))$. In addition, there are two other steady states, located at $(e_t, \kappa_t) = (e(\kappa_t), \kappa_t)$. These are given by the points of intersection between the XX and EE locus. The region with both positive growth rates of education and capital per effective worker always converges to a positive output growth rate across all regimes. Special attention is needed for the orange region, particularly in the Malthusian regime. This region corresponds to cases of negative growth in either capital per effective worker, education, or both. Economies in this region converge to the Conditional Malthusian steady state, where $e_t = 0$, $\kappa = z^{-1}(\zeta)$, $y_t = (1-\alpha)(z{-}1(\zeta))\,\alpha$, and $\frac{\dot{y}}{y} = 0$. This outcome, labeled as **ML** trap (a situation which can arise due to rising growth rate of population), signifies zero output growth, making the transition from the Malthusian to the Post-Malthusian and **MG** regimes impossible. The orange region in the **MG** regime indicates that, as education (e_t) evolves and technological

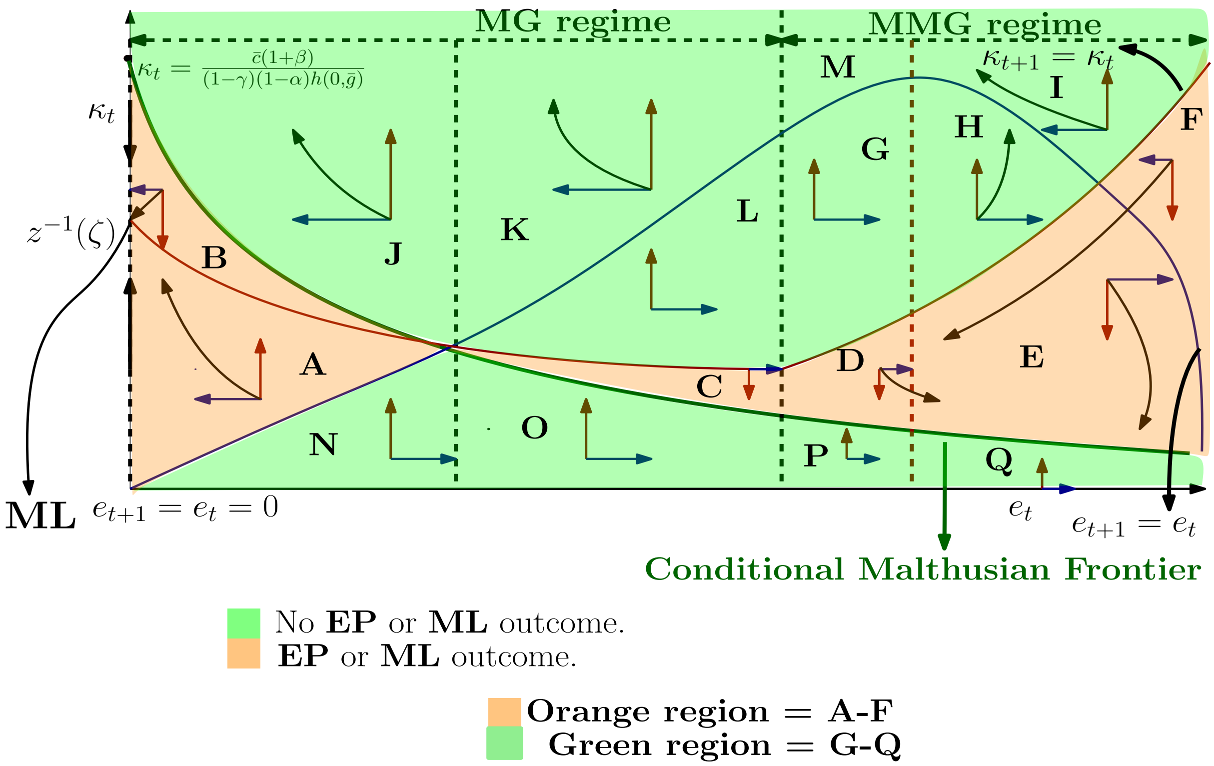


Figure 6: The conditional dynamic system without inequality .

progress advances, the population growth rate (g_{t+1}) starts to decline. Over time, this declining growth rate dominates, leading to a slowdown in technological progress and ultimately resulting in an **EP** outcome (an outcome which can arise due to declining growth rate of population). If the economy transitions to the orange region in the **MMG** regime, it may either converge to a Conditional Malthusian steady state (**ML**) trap, or it may achieve a region of positive output growth. However, if capital per effective labor declines due to population increase, the economy risks being trapped in a Malthusian regime again, potentially repeating its history. In conclusion, while education and technological progress are influential, they may not ensure sustained output growth or fully explain the transition between the Malthusian, **MG**, and **MMG** regimes.

6.2 Dynamical system with inequality

To analyze how the evolution of education influences inequality through technological progress, we now consider households with varying levels of endowed human capital. If the income distribution positions the economy within the Malthusian regime, that is, the relative human capital, a_t, is uniformly distributed within the range, $\frac{\bar{c}}{w_t\bar{h}_t} \leq a_t \leq \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t}$, then we have,

$$K_{t+1} = P_t * A, \tag{22}$$

$$L_t = \frac{P_t(1+\beta)}{w_t \gamma \bar{c}} \left[f(w_t \bar{h}_t = z_t, \eta(g(e_t; P))) \right], \tag{23}$$

$$\kappa_{t+1} = \Xi^a(e_t, g_t, \kappa_t; P)\kappa_t, \tag{24}$$

$$e_{t+1} = \Phi^a(e_t, g_t, \kappa_t; P). \tag{25}$$

During this regime, technological progress increases the returns to education (η) and investment in education (e_{t+1}), leading to a rise in overall fertility (n_t) as the income effect outweighs the substitution effect. This relationship is characterized by $\Xi^a_\kappa(.) < 0$, $\Xi^a_e(.) > 0$, and $\frac{d\kappa_t}{de_t} > 0$. Simultaneously, with technological progress, $\Phi^a_\kappa(.) < 0$ and $\Phi^a_e(.) < 0$ as the income effect continues to dominate, implying $\frac{d\kappa}{de_t} < 0$. Technological progress during this period also exacerbates inequality in the subsequent period ($t+1$). Consequently, fertility (n_{t+1}) and investment in children's education (e_{t+2}) further rise during the Malthusian regime, as highlighted in Proposition 2. This is reflected in $\Xi^a_e(.) > 0$, $\frac{d\kappa_t}{de_t} > 0$, and $\Phi^a_e(.) = 0$, causing a rightward shift in the e curve. As education evolves, the XX locus is upward-sloping due to the influence of technological progress and inequality, while the EE locus is downward-sloping with technological progress but shifts upward in response to increased inequality.

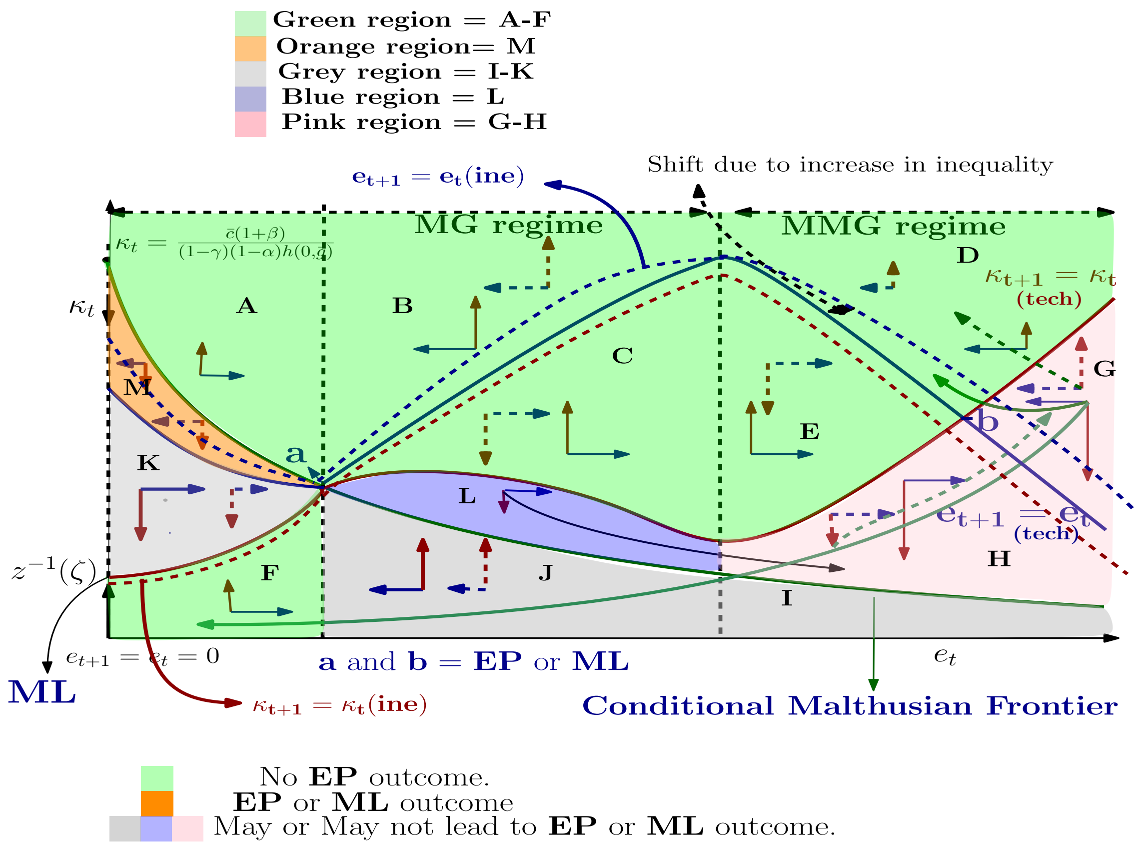


Figure 7: The conditional dynamic system with inequality.

If the income distribution places the economy within the **MG** regime, where relative human capital a_t is uniformly distributed as $\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t} \leq a_t \leq (w_t\bar{h}_t)\left(\frac{w_t\bar{h}_t\theta-\psi}{\psi}\right)$, then we have,

$$K_{t+1} = P_t * B(w_t \bar{h}_t), \tag{26}$$
$$\kappa_{t+1} = \Xi^c(e_t, g_t, \kappa_t; P),$$
$$e_{t+1} = \Phi^b(z_t, \kappa_t, g_t; P).$$

During the **MG** regime, technological progress leads to rising returns to education (η) and investment in education (e_{t+1}), while overall fertility (n_t) declines as the substitution effect outweighs the income effect. This is reflected in $\Xi^c_\kappa(.) > 0$, $\Xi^c_e(.) > 0$, and $\frac{d\kappa_t}{de_t} < 0$. Simultaneously, with technological progress, $\Phi^b_\kappa(.) < 0$ and $\Phi^b_e(.) > 0$, indicating the dominance of the substitution effect, which implies $\frac{d\kappa}{de_t} > 0$. As previously noted, technological progress also contributes to greater inequality in period $t + 1$. Consequently, fertility (n_{t+1}) and average investment in children's education (e_{t+2}) decline during the **MG** regime, as observed in Proposition 2. This results in $\Xi^c_e(.) < 0$, $\frac{d\kappa_t}{de_t} > 0$, and $\Phi^b_e(.) = 0$, causing a leftward shift in the e curve. As education evolves, the XX locus becomes downward sloping due to technological progress and upward sloping due to inequality. Meanwhile, the EE locus is upward sloping with technological progress but shifts leftward in response to increasing inequality.

If the income distribution places the economy within the **MMG** regime, where relative human capital, a_t, is uniformly distributed as $(w_t \bar{h}_t)\left(\frac{w_t \bar{h}_t \theta - \psi}{\psi}\right) \leq a_t$, then we have,

$$K_{t+1} = P_t * C(w_t \bar{h}_t), \tag{27}$$
$$\kappa_{t+1} = \Xi^d(e_t, g_t, \kappa_t; P),$$
$$e_{t+1} = \Phi^c(e_t, g_t, \kappa_t; P).$$

Technological progress results in higher returns to education (η) and greater investment in education (e_{t+1}), accompanied by a rise in overall fertility (n_t) due to the income effect dominating the substitution effect. This relationship is expressed as $\Xi^d_\kappa(.) > 0$, $\Xi^d_e(.) < 0$, and $\frac{d\kappa_t}{de_t} > 0$. At the same time, technological progress results in $\Phi^c_\kappa(.) < 0$ and $\Phi^c_e(.) < 0$, as the income effect continues to dominate, leading to $\frac{d\kappa}{de_t} < 0$. With rising inequality during the **MMG** regime, both fertility (n_{t+1}) and investment in children's education (e_{t+2}) increase, as highlighted in Proposition 2. This outcome is characterized by $\Xi^a_e(.) > 0$, $\frac{d\kappa_t}{de_t} < 0$, and $\Phi^a_e(.) = 0$, resulting in an upward shift in the e curve. These dynamics reflect that the XX locus slopes upward with technological progress but slopes downward with rising inequality. On the other hand, the EE locus is downward-sloping with technological progress but shifts upward with increased inequality.

Figure (7) illustrates the pivotal role of technological progress and inequality in driving the economic transition from the Malthusian regime to the **MG** regime, and subsequently to the **MMG** regime. As education improves, inequality and technological progress increase, altering economic outcomes. The dotted lines and arrows indicate the effects of rising inequality, while the bolder lines and arrows represent the impact of increased technological progress on capital per effective labor (κ) and education investment (e). Below the Conditional Malthusian Frontier (green curve), households are limited to subsistence-level consumption. Technological progress alone is insufficient to drive the transition from the Malthusian to the **MG** regime. Figure (6) demonstrates that, without considering inequality, as education evolves and technological progress increases, an economy could converge to a Conditional Malthusian steady state, that is, **ML** trap (orange region). Introducing inequality shows that when its effects surpass those of technological progress in the absence of inequality, economies in the grey region show transition from the Malthusian to the **MG** regime. This shift eliminates the orange region, moves the grey region upward, and fosters economic growth. Inequality thus plays a more crucial role in this transition. In the dark blue region, both inequality and technological progress together lead to either an *EP* outcome or an economy may move to **MMG** regime. Upon entering the **MMG** regime (pink region), technological progress may return the economy to the **Malthusian** region, causing another **ML** trap. Despite this, rising inequality plays a crucial role in preserving a positive growth rate and averting this regression to **ML** trap, thereby reinforcing its dominant influence in achieving long-term economic growth and development. Combining subsection (6.1) and (6.2), we present the following proposition.

Proposition 3. ***EP**, as outlined in Jones (2022), is a possible outcome, though economies may relapse into the **ML** trap, repeating historical patterns. Technological progress alone cannot drive the fertility transition or ensure sustained output growth; instead, enduring growth depends on how technological advances shape and interact with inequality.*

Proof 3. *The proof follows directly from above and Appendix A.2.*

Negative population growth leads to an **EP** outcome. With constant investment in education and a declining population growth rate, the stock of ideas decreases, i.e., $\frac{\dot{A}_t}{A_t} < 0$. This aligns with negative technological progress represented by $g_{t+1} < 0$. Such a scenario arises when the negative impact of population decline outweighs the positive influence of rising education, resulting in the persistence of the **EP** outcome.

During the **MG** regime, where $\frac{\dot{e}}{e} \geq 0$ and $\frac{\dot{n}}{n} < 0$, if $\frac{\dot{e}}{e} = 0$ and $\frac{\dot{n}}{n} < 0$, the economy consistently converges to the **EP** outcome. This occurs because education contributes nothing to the stock of ideas while fertility decline has a negative effect, resulting in $g_{t+1}(e_t, P_t) < 0$. In cases of a quantity-quality trade-off, where $\frac{\dot{e}}{e} > 0$ and $\frac{\dot{n}}{n} < 0$, the economy reaches the **EP** outcome if the negative population growth ($\frac{\dot{n}}{n}$) surpasses the positive educational effect ($\frac{\dot{e}}{e}$). This leads to declining technological progress,

$g_{t+1}(e_t, P_t) < 0$, reinforcing the **EP** outcome. In summary, we show that despite advancements in education and technological progress, as well as a decline in fertility and population growth, the dominance of negative population effects over rising education drives the economy toward the **EP** outcome during the **MG** regime. When an economy moves from the **MG** regime to the **MMG** regime, fertility begins to rise alongside potential income. Technological progress, coupled with the effects of inequality, can then propel the economy onto a path of sustained growth.

7 Computation

The theoretical findings in the previous section highlight several crucial aspects. First, we examine how inequality in human capital drives disparities in fertility and education, subsequently influencing economic growth. Second, historical fertility trends can jointly be explained by technological progress and its effect on inequality. Third, we analyze whether a declining population growth rate results in stagnant economic growth, represented as the **EP** outcome. The model assumes a generational length of 30 years for calibration. The calibration process is divided into two parts: the first considers an exogenous level of returns to education, where η remains fixed, following de La Croix and Doepke (2003). The second part involves a comprehensive calibration of our unified model across the entire period, incorporating an endogenous level of returns to education, where η is a function of technological progress, following Galor and Weil (2000). The following table contains the values of the parameters taken from de la Croix and Doepke (2003), Bhattacharya and Chakroborty (2007) and Cordoba and Ripoll (2019).

Parameters	Value	Source
α	0.33	de la Croix and Doepke (2003)
β	0.37	de la Croix and Doepke (2003), Bhattacharya and Chakroborty (2007)
θ	0.0118	de la Croix and Doepke (2003)
γ	0.271	de la Croix and Doepke (2003)
ϕ	0.075	de la Croix and Doepke (2003), Cordoba and Ripoll (2019)
ψ	0.001	Bhattacharya and Chakroborty (2007)
δ	0.015	Bhattacharya and Chakroborty (2007)
η	0.635	de la Croix and Doepke (2003)

The parameter ϕ represents the time cost of raising a child, computed by de la Croix and Doepke (2003) and Cordoba and Ripoll (2019) from age 0 to 17, varying across income levels. On average, this time cost is approximately 7.5%. Focusing on growth, ϑ is set to $1-\tau$, where τ indicates the direct effect of parental human capital on the child's human capital. Following de la Croix and Doepke (2003), τ is assumed to be 0.2.

7.1 Initial inequality, fertility and growth

In our first computational experiment, we examine how initial inequality affects human capital growth under the assumption of a fixed level of returns to education, i.e, starting with initial level of $\eta = 0.635$. By excluding the lifetime subsistence consumption constraint, Figure (8) demonstrates that as inequality increases, human capital growth declines. The blue line corresponds with the findings of de la Croix and Doepke (2003) (orange line), as their analysis also assumed a fixed level of returns to education and excluded the Malthusian regime. Consequently, the blue line becomes obscured due to its overlap with the orange line. After incorporating the Malthusian regime, Figure (9) illustrates that human capital growth initially rises with increasing inequality but ultimately declines. This pattern suggests that during the Malthusian regime, inequality fosters human capital growth; however, in the **MG** regime, further inequality has a suppressive effect on it. Figure (10) illustrate the trajectories of human capital, fertility, education, capital, GDP, inequality, and differential fertility over 240 years ($t = 1$ to $t = 8$), considering the Malthusian regime and fixed level of η. With a decrease in inequality, total population declines and eventually turns negative, while capital, human capital growth, and GDP growth rises, contradicting the **EP** outcome. With fixed level of η, population decline indirectly boosts human capital growth via education and directly raises GDP growth due to fewer individuals, without any adverse effects on these variables. However, if η is assumed to be a function of technological progress, which itself depends on total population and education, a negative population growth directly reduces human capital growth through declining technological progress and returns to education (η). This would align the economy with the **EP** outcome under negative population growth, as shown by Jones (2022).

7.2 With technological progress

We now turn to dynamic implication of our model. We have assumed returns to education, η, as strictly concave and increasing function of technological progress which itself is strictly concave and increasing function of education and total population. We adhere to the following specific form of return to education:

$$\eta = a + (g_{t+1}(e_t, P_t))^{\varepsilon} \equiv a + \left((P_t)^{\zeta}(e_t)^{1-\zeta}\right)^{\varepsilon}.$$

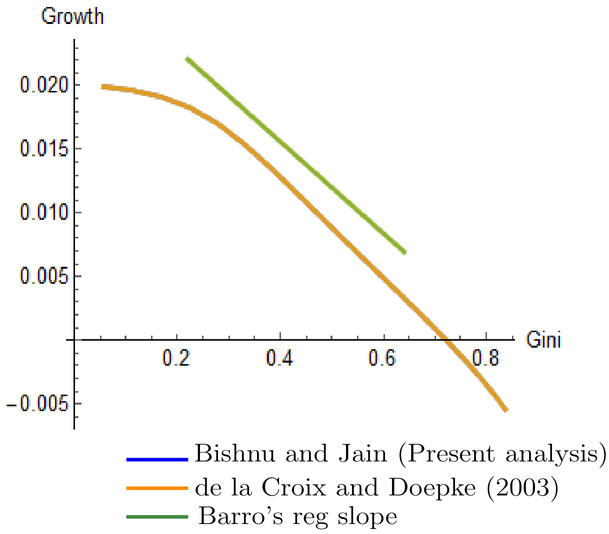


Figure 8: Gini vs Growth with no Malthusian regime and exogenous $\eta = 0.635$.

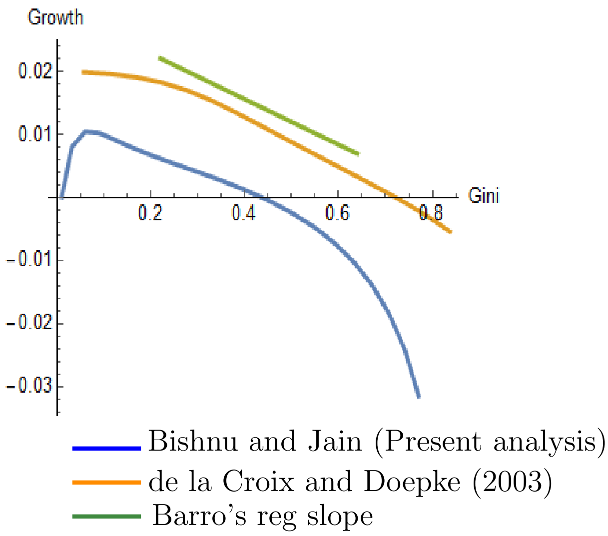


Figure 9: Gini vs Growth with Malthusian regime and exogenous $\eta = 0.635$.

We have chosen the parameter values, $\gamma = 0.01$, $\zeta = 0.65$ and $\varepsilon = 0.33$ such that the elasticity of human capital with respect to education lies with in the acceptable range of values mentioned in the literature, that is $\eta \in [0.635, 0.7]$. Figure (11) confirms our theoretical claim that fertility initially rises, then declines, and eventually starts rising with income. The *income-fertility* curve for 21 OECD countries, presented in **Appendix** B.5, also illustrates this trend of initial increase, subsequent decline, and slight rise. Figure (12) illustrates the impact of inequality on total population, education, and economic growth. Figure (12a) shows that if an economy starts with Malthusian regime that is initial level of relative human capital, $a_t = 0.21$, then rising inequality leads to an increase in population and education growth. However, as economy transitions into the **MG** regime, that is initial level of relative human capital, $a_1 = 1$ as in Figure (12b) population and education growth begins to decline. GDP growth, on the other hand,

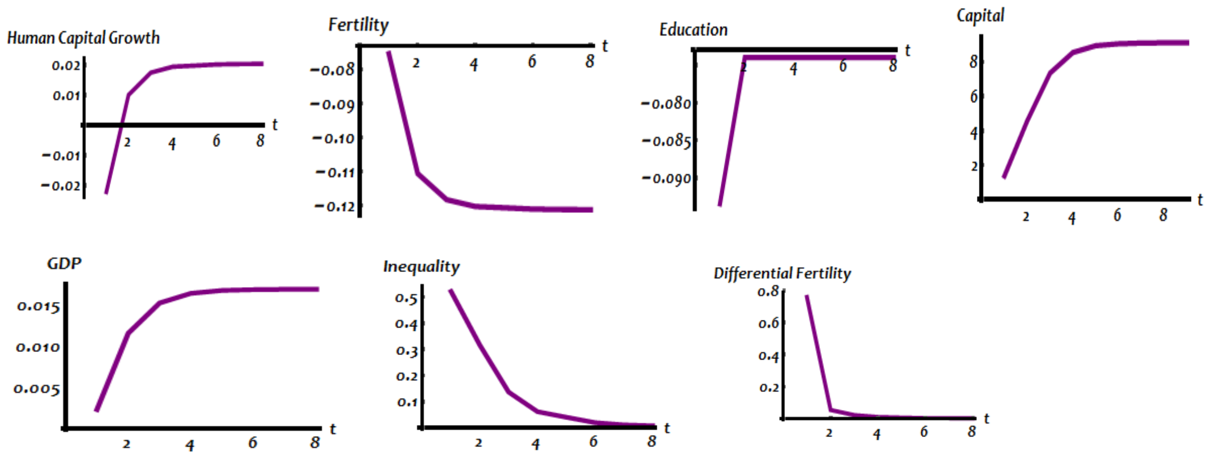


Figure 10: Human capital, Fertility, Education, Capital, GDP, Inequality, Differential fertility.

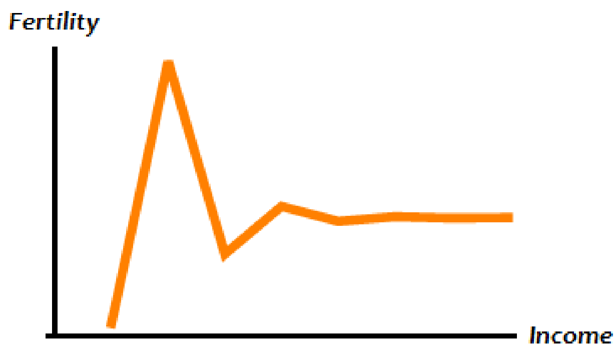
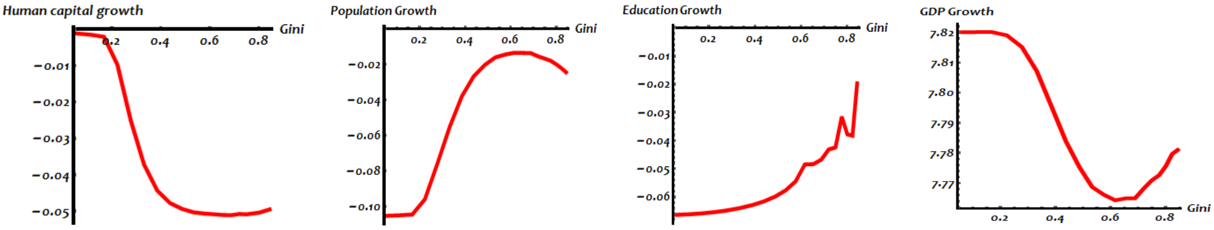
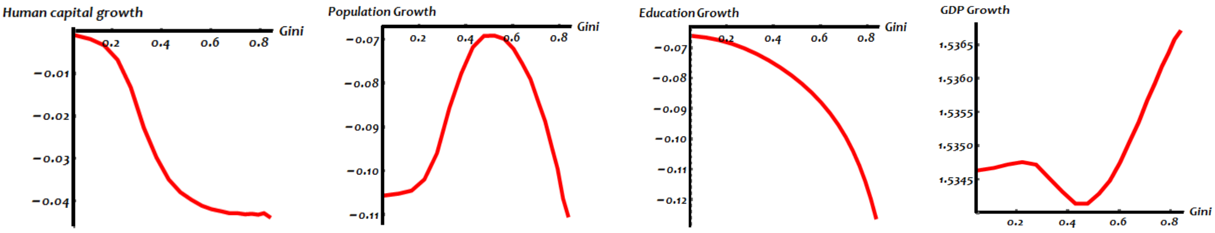


Figure 11: Income-fertility path.

initially falls but eventually starts rising. This supports our claim that the influence of inequality on fertility and education is contingent on the regime in which an economy lies. Figure (13) shows how inequality and technological progress together can explain historical fertility path (green curve). We can see that from period $t = 1$ to $t = 3$, with decrease in inequality and increase in technological progress, total population falls. As during Malthusian regime decrease in inequality leads to decrease in fertility and with increase in technological progress fertility rises as income effect dominates substitution effect; overall fertility falls if inequality effect dominates the impact of technological progress. As during the **MG** regime with rise in technological progress and inequality, fertility rises and education falls during period $t = 4$ to $t = 5$. During this regime, rising inequality leads to a drop in education levels. If the impact of inequality outweighs technological advancements, education may decline despite technological growth. Together, inequality and technological progress shape the trajectory of fertility



(a) Initial level of relative human capital, $a_t = 0.21$



(b) Initial level of relative human capital, $a_t = 1$

Figure 12: Gini vs Growth with endogenous η.

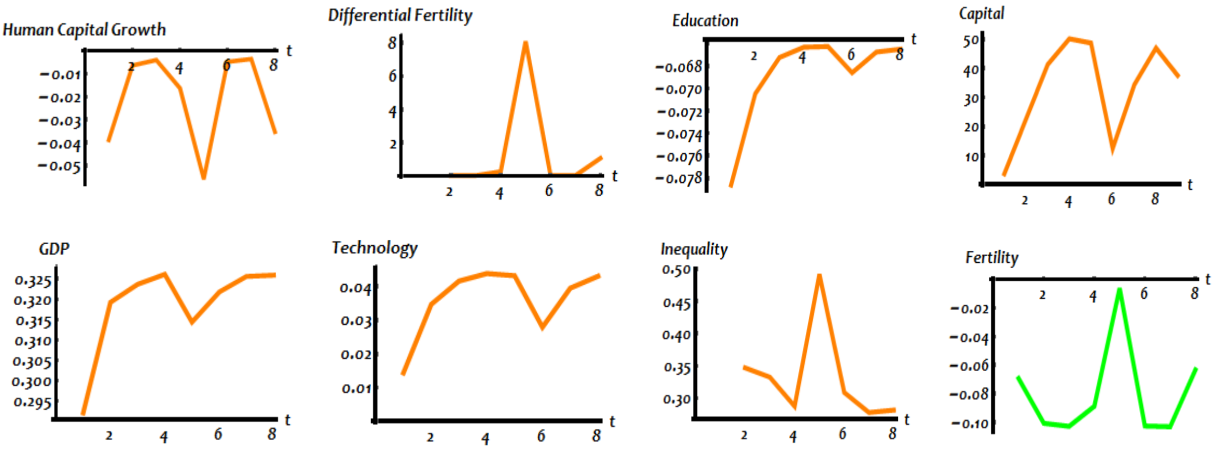


Figure 13: Human capital, Fertility, Education, Capital, GDP, Inequality, Differential fertility.

and economic growth. It is evident that between periods $t = 7$ and $t = 8$, increasing population growth combined with constant education growth leads to an increase in technological growth but inequality shows a declining trend. Consequently, GDP growth begin to stagnate, leading the economy into an **ML** outcome as the effect of rising population outpaces the effect of rising education. By altering parameter values and initial conditions, a variety of fertility trajectories can be generated, reflecting the significant variations observed across countries due to their unique starting points. Through computational exercises involving changes to these parameter values, we identified the possibility of an economy moving to an **EP** outcome.

8 Conclusion

We conduct a comprehensive analysis demonstrating how inequality has historically shaped the *income-fertility* relationship, driven fertility transitions over time from the Malthusian era to the present, and will likely influence future trends. In our framework, inequality operates through channels of knowledge diffusion and technological advancement. The *income-fertility* trajectory rises during the Malthusian regime, declines in the **MG** regime, and exhibits a potential slight increase at very high income levels within the **MMG** regime - a pattern recently observed in the United States. Contrary to existing literature, we find that rising inequality reduces fertility during the **MG** regime but increases it during the Malthusian and **MMG** regimes. Technological progress consistently reduces fertility across all regimes by enhancing education, reflecting a substitution effect. While prior research suggests that fertility initially rises in the Malthusian regime as income effects dominate substitution effects, we show this pattern is not universal, as it overlooks the concurrent rise in inequality driven by technological change.

We further explore the global dynamics of future economic states based on the evolution of inequality and technological growth, including the possibility of **EP**-type scenarios. We find that an **EP** outcome - characterized by negative population growth and declining technological progress - may occur within the **MG** regime if fertility falls sharply. Transitioning into the **MMG** regime, where fertility rises with income, could reintroduce Malthusian-like stagnation. Economies with low initial capital per effective labor and rising fertility are particularly vulnerable to a Malthusian-like (**ML**) trap. However, rising inequality, by simultaneously boosting fertility and education, can facilitate escape from this trap, enabling positive output growth. In conclusion, our framework bridges a critical gap in the literature by providing a unified, empirically grounded explanation of the income-fertility relationship and fertility transitions through the joint lens of inequality and technological change.

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Online Appendices

A Appendix A

A.1 Proof of Proposition 1

All the derivations has been shown in **Appendix B**, the optimal choice of fertility, n_t, is given by,

$$n_t = \begin{cases} \frac{\left(a_t w_t \bar{h}_t - \bar{c}\right)}{\left(a_t \phi w_t \bar{h}_t + \psi\right)} & \text{if } \frac{\bar{c}}{w_t \bar{h}_t} \leq a_t < \frac{\theta}{(\phi\eta(g_{t+1}) - \theta\delta)} - \frac{\eta\psi}{(\phi\eta(g_{t+1}) - \theta\delta) w_t \bar{h}_t}, \\ \frac{\left(a_t w_t \bar{h}_t - \bar{c}\right)(1 - \eta(g_{t+1}))}{\left(a_t \phi w_t \bar{h}_t + \psi - w_t \bar{h}_t \theta(1 + \delta a_t)\right)} & \text{if } \frac{\theta}{(\phi\eta(g_{t+1}) - \theta\delta)} - \frac{\eta\psi}{(\phi\eta(g_{t+1}) - \theta\delta) w_t \bar{h}_t} \leq a_t < \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta) w_t \bar{h}_t}, \\ \frac{\gamma(1 - \eta(g_{t+1})) a_t w_t \bar{h}_t}{(1+\beta+\gamma)\left[a_t \phi w_t \bar{h}_t + \psi - w_t \bar{h}_t \theta(1 + \delta a_t)\right]} & \text{if } \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta) w_t \bar{h}_t} \leq a_t. \end{cases} \tag{A.28}$$

Using (A.28) we can check, fertility increases with an increase in fixed income, that is,

$$\frac{dn_t}{d(w_t \bar{h}_t)} > 0 \ \ \forall a_t \geq \frac{\bar{c}}{w_t \bar{h}_t}, \tag{A.29}$$

whereas the relationship between fertility and relative human capital is given by,

$$\frac{dn_t}{da_t} \begin{cases} > 0, & \text{if } \frac{\bar{c}}{w_t \bar{h}_t} \leq a_t \leq \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta) w_t \bar{h}_t}, \\ < 0, & \text{if } a_t \geq \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta) w_t \bar{h}_t}. \end{cases} \tag{A.30}$$

Therefore, using (A.29) and (A.30) we see that,

$$\frac{dn_t}{da_t} \text{ and } \frac{dn_t}{d(w_t \bar{h}_t)} > 0 \quad \text{if} \quad \frac{\bar{c}}{w_t \bar{h}_t} \leq a_t \leq \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta) w_t \bar{h}_t},$$

$$\frac{dn_t}{da_t} = \frac{\gamma(1-\eta) w_t \bar{h}_t (\psi - w_t \bar{h}_t \theta)}{(1+\beta+\gamma)\left[a_t \phi w_t \bar{h}_t + \psi - w_t \bar{h}_t \theta(1 + \delta a_t)\right]^2} < 0 \quad \text{if} \quad a_t \geq \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta) w_t \bar{h}_t},$$

$$\frac{dn_t}{d(w_t \bar{h}_t)} = \frac{\gamma(1-\eta) a_t \psi}{(1+\beta+\gamma)\left[a_t \phi w_t \bar{h}_t + \psi - w_t \bar{h}_t \theta(1 + \delta a_t)\right]^2} > 0 \quad \text{if} \quad a_t \geq \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta) w_t \bar{h}_t}.$$

Hence, fertility increases or decreases with potential income depends on the dominance of two effects - $w_t\bar{h}_t$(fixed income) and a_t(relative human capital),

$$\frac{dn_t}{d(w_t h_t)}\begin{cases}\frac{dn_t}{da_t}+\frac{dn_t}{d(w_t\bar{h}_t)}>0, & \text{if } \frac{\bar{c}}{w_t\bar{h}_t}\leq a_t\leq\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t},\\ \frac{dn_t}{da_t}+\frac{dn_t}{d(w_t\bar{h}_t)}<0, & \text{if } \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t}\leq a_t\leq\frac{w_t\bar{h}_t(w_t\bar{h}_t\theta-\psi)}{\psi},\\ \frac{dn_t}{da_t}+\frac{dn_t}{d(w_t\bar{h}_t)}>0, & \text{if } a_t\geq\frac{w_t\bar{h}_t(w_t\bar{h}_t\theta-\psi)}{\psi}.\end{cases}$$

Therefore, we can see fertility initially increases thereafter it declines and then it starts rising with potential income.

A.2 Proof of Proposition 3

During the **MG** regime, when fertility is declining with potential income, that is the region with $\frac{\bar{c}(1+\beta)}{(1+\beta+\gamma)w_t\bar{h}_t}<a_t<(w_t\bar{h}_t)\left(\frac{w_t\bar{h}_t\theta-\psi}{\psi}\right)$,

$$n_t=\frac{\gamma(1-\eta)a_t w_t\bar{h}_t}{(1+\beta+\gamma)(a_t\phi w_t\bar{h}_t+\psi-w_t\bar{h}_t\theta(1+\delta a_t))}.$$

Assuming no inequality, i.e., each household is endowed with same level of relative human capital, $a_t=a$ then population growth turns negative when $n_t-1<0$, that is,

$$w_t(\kappa_t)h_t<\frac{(1+\beta+\gamma)\psi}{\gamma(1-\eta)a+(1+\beta+\gamma)\theta(1+\delta a)-(1+\beta+\gamma)a\phi}.$$

Hence, with low level of capital per effective labor, population growth turns negative. The optimal level of investment in education and growth rate of education is given by,

$$e_{t+1}=\frac{\eta(aw_t h_t\phi+\psi)-w_t h_t\theta(1+\delta a)}{(1-\eta)w_t h_t(1+\delta a)},$$

$$\dot{e}_{t+1}=\frac{\frac{1}{1-\eta}\left(\frac{d\eta}{dt}\left[(a\phi w_t\bar{h}_t+\psi-w_t\bar{h}_t\theta(1+\delta a))\right]\right)-\frac{d(w_t\bar{h}_t)}{dt}\left(\frac{\eta\psi}{w_t\bar{h}_t}\right)+\frac{da}{dt}\left(\eta\phi w_t\bar{h}_t-\eta\delta\psi\right)}{(1-\eta)(1+\delta a)(w_t\bar{h}_t)}>0.$$

The growth rate of education increases with a rise in relative human capital and returns to education, but it decreases with high fixed income. During the **MG** regime, the impact of the first two factors outweighs the latter. Consequently, the economy transitions to the **EP** outcome when the growth rate of education is positive, but population growth is negative. Technological progress begins to decline as the negative population growth rate's effect surpasses the positive impact of the rising education growth rate.

B Appendix B

B.1 Derivation of equations (9)-(10)

The optimization problem is given by,

$$argmax\{ln(c_t) + \beta ln(s_t R_{t+1}) + \gamma ln(w_{t+1} h_{t+1} n_t)\},$$

s.t.

$$c_t + \frac{d_{t+1}}{R_{t+1}} = w_t \bar{h}_t (a_t(1 - (\phi + \delta e_{t+1})n_t) - e_{t+1} n_t) - \psi n_t \geq \bar{c},$$

and,

$$\{n_t, e_{t+1}\} \geq 0.$$

For households with $c_t + s_t = \bar{c}$, the optimal choice of fertility is given by,

$$n_t = \frac{a_t w_t \bar{h}_t - \bar{c}}{a_t(\phi + \delta e_{t+1}) w_t \bar{h}_t + e_{t+1} w_t \bar{h}_t + \psi}. \tag{B.31}$$

The optimal choice of saving is given by,

$$\frac{dU_t}{ds_t} = -\frac{1}{c_t} + \frac{\beta}{s_t} = 0,$$

hence

$$s_t = \frac{\beta \bar{c}}{1 + \beta},$$

precisely

$$s_t = \begin{cases} 0, & \text{if } a_t \leq \frac{\bar{c}}{w_t \bar{h}_t} \\ \frac{\beta \bar{c}}{(1+\beta)}, & \text{if } \frac{\bar{c}}{w_t \bar{h}_t} \leq a_t < \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta) w_t \bar{h}_t}. \end{cases}$$

The optimal choice of investment in child's education is given by,

$$\frac{dU_t}{de_{t+1}} = \frac{\gamma}{n_t} \frac{dn_t}{de_{t+1}} + \frac{\gamma}{h_{t+1}} \frac{dh_{t+1}}{de_{t+1}} = 0,$$

$$\frac{dU_t}{de_{t+1}} = -\frac{\gamma w_t \bar{h}_t (1 + \delta a_t)}{a_t(\phi + \delta e_{t+1}) w_t \bar{h}_t + e_{t+1} w_t \bar{h}_t + \psi} + \frac{\gamma \eta}{\theta + e_{t+1}} = 0,$$

$$e_{t+1} = \frac{\eta(a_t \phi w_t \bar{h}_t + \psi) - w_t \bar{h}_t \theta (1 + \delta a_t)}{(1 - \eta)(1 + \delta a_t) w_t \bar{h}_t}. \tag{B.32}$$

Substituting (B.32) in (B.31) we get,

$$n_t = \frac{(1-\eta)(a_t w_t \bar{h}_t - \bar{c})}{(a_t \phi w_t \bar{h}_t + \psi - w_t \bar{h}_t \theta(1+\delta a_t))}. \tag{B.33}$$

The threshold level of relative human capital for households with $c_t + s_t = \bar{c}$ is,

$$\frac{dU_t}{dn_t} = \frac{1}{c_t}\frac{dc_t}{dn_t} + \frac{\gamma}{n_t} > 0,$$

$$\frac{dU_t}{dn_t} = -\frac{1}{\bar{c} - s_t} + \frac{\gamma}{(a_t w_t \bar{h}_t - \bar{c})} > 0,$$

$$a_t < \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta) w_t \bar{h}_t}. \tag{B.34}$$

The choice of e_{t+1} is positive if,

$$a_t > \frac{\theta}{\phi\eta - \theta\delta} - \frac{\eta\psi}{(\phi\eta - \theta\delta) w_t \bar{h}_t}. \tag{B.35}$$

Now, we have three main cases to consider,

i) Case 1: $\frac{\theta}{\phi\eta-\theta\delta} - \frac{\eta\psi}{(\phi\eta-\theta\delta)w_t\bar{h}_t} < \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t}$, i.e., $w_t\bar{h}_t < \frac{\phi\eta-\theta\delta}{\theta}\left[\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)} + \frac{\eta\psi}{\phi\eta-\theta\delta}\right]$,

ii) knife edge case: $\frac{\theta}{\phi\eta-\theta\delta} - \frac{\eta\psi}{(\phi\eta-\theta\delta)w_t\bar{h}_t} = \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t}$, i.e., $w_t\bar{h}_t = \frac{\phi\eta-\theta\delta}{\theta}\left[\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)} + \frac{\eta\psi}{\phi\eta-\theta\delta}\right]$,

iii) Case 2: $\frac{\theta}{\phi\eta-\theta\delta} - \frac{\eta\psi}{(\phi\eta-\theta\delta)w_t\bar{h}_t} > \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t}$, i.e., $w_t\bar{h}_t > \frac{\phi\eta-\theta\delta}{\theta}\left[\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)} + \frac{\eta\psi}{\phi\eta-\theta\delta}\right]$.

Using (B.31), (B.32), (B.33), (B.34) and (B.35), the optimal choice of fertility and investment in child's education is given by,

Case 1: $w_t\bar{h}_t < \frac{\phi\eta-\theta\delta}{\theta}\left[\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)} + \frac{\eta\psi}{\phi\eta-\theta\delta}\right]$,

$$n_t = \begin{cases} 0, & \text{if } a_t \le \frac{\bar{c}}{w_t\bar{h}_t} \\ \frac{(a_t w_t\bar{h}_t - \bar{c})}{(a_t\phi w_t\bar{h}_t + \psi)}, & \text{if } \frac{\bar{c}}{w_t\bar{h}_t} \le a_t < \frac{\theta}{\phi\eta-\theta\delta} - \frac{\eta\psi}{(\phi\eta-\theta\delta)w_t\bar{h}_t} \\ \frac{(1-\eta)(a_t w_t\bar{h}_t - \bar{c})}{(a_t\phi w_t\bar{h}_t + \psi - w_t\bar{h}_t\theta(1+\delta a_t))}, & \text{if } \frac{\theta}{\phi\eta-\theta\delta} - \frac{\eta\psi}{(\phi\eta-\theta\delta)w_t\bar{h}_t} \le a_t < \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t}, \end{cases}$$

$$e_{t+1} = \begin{cases} 0, & \text{if } a_t \le \frac{\bar{c}}{w_t\bar{h}_t} \\ 0, & \text{if } \frac{\bar{c}}{w_t\bar{h}_t} \le a_t < \frac{\theta}{\phi\eta-\theta\delta} - \frac{\eta\psi}{(\phi\eta-\theta\delta)w_t\bar{h}_t} \\ \frac{\eta(a_t\phi w_t\bar{h}_t + \psi) - w_t\bar{h}_t\theta(1+\delta a_t)}{(1-\eta)(1+\delta a_t)w_t\bar{h}_t}, & \text{if } \frac{\theta}{\phi\eta-\theta\delta} - \frac{\eta\psi}{(\phi\eta-\theta\delta)w_t\bar{h}_t} \le a_t < \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t}. \end{cases}$$

We can see that, $\frac{dn_t}{da_t} \geq 0$, $\frac{de_{t+1}}{da_t} \geq 0$, $\frac{d^2 n_t}{da_t^2} \leq 0$ and $\frac{d^2 e_{t+1}}{da_t^2} \leq 0$.

Knife edge case: $w_t \bar{h}_t = \frac{\phi\eta - \theta\delta}{\theta} \left[\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)} + \frac{\eta\psi}{\phi\eta - \theta\delta} \right]$,

$$n_t = \begin{cases} 0, & \text{if } a_t \leq \frac{\bar{c}}{w_t \bar{h}_t} \\ \frac{(a_t w_t \bar{h}_t - \bar{c})}{(a_t \phi w_t \bar{h}_t + \psi)}, & \text{if } \frac{\bar{c}}{w_t \bar{h}_t} \leq a_t < \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta) w_t \bar{h}_t}, \end{cases}$$

$$e_{t+1} = \begin{cases} 0, & \text{if } a_t \leq \frac{\bar{c}}{w_t \bar{h}_t} \\ 0, & \text{if } \frac{\bar{c}}{w_t \bar{h}_t} \leq a_t < \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta) w_t \bar{h}_t}. \end{cases}$$

So we can see that, $\frac{dn_t}{da_t} \geq 0$, $\frac{de_{t+1}}{da_t} = 0$, $\frac{d^2 n_t}{da_t^2} \leq 0$ and $\frac{d^2 e_{t+1}}{da_t^2} = 0$.

Case 2: $w_t \bar{h}_t > \frac{\phi\eta - \theta\delta}{\theta} \left[\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)} + \frac{\eta\psi}{\phi\eta - \theta\delta} \right]$,

$$n_t = \begin{cases} 0, & \text{if } a_t \leq \frac{\bar{c}}{w_t \bar{h}_t} \\ \frac{(a_t w_t \bar{h}_t - \bar{c})}{(a_t \phi w_t \bar{h}_t + \psi)}, & \text{if } \frac{\bar{c}}{w_t \bar{h}_t} \leq a_t < \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta) w_t \bar{h}_t}, \end{cases}$$

$$e_{t+1} = \begin{cases} 0, & \text{if } a_t \leq \frac{\bar{c}}{w_t \bar{h}_t} \\ 0, & \text{if } \frac{\bar{c}}{w_t \bar{h}_t} \leq a_t < \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta) w_t \bar{h}_t}. \end{cases}$$

So we can see that, $\frac{dn_t}{da_t} \geq 0$, $\frac{de_{t+1}}{da_t} = 0$, $\frac{d^2 n_t}{da_t^2} \leq 0$ and $\frac{d^2 e_{t+1}}{da_t^2} = 0$.

For households with $c_t + s_t > \bar{c}$, the optimization problem is,

$$argmax\{ln(c_t) + \beta ln(d_{t+1}) + \gamma(w_{t+1} h_{t+1} n_t)\},$$

s.t.

$$c_t + \frac{d_{t+1}}{R_{t+1}} = w_t \bar{h}_t (a_t (1 - (\phi + \delta e_{t+1}) n_t) - e_{t+1} n_t) - \psi n_t > \bar{c},$$

and,

$$\{n_t, e_{t+1}\} \geq 0.$$

The optimal choice of saving is given by,

$$\frac{dU_t}{ds_t} = -\frac{1}{c_t} + \frac{\beta}{s_t} = 0,$$

$$s_t = \beta c_t. \tag{B.36}$$

The optimal choice of fertility is given by,

$$\frac{dU_t}{dn_t} = \frac{1}{c_t}\frac{dc_t}{dn_t} + \frac{\gamma}{n_t} = 0,$$

$$\frac{\gamma}{n_t} = \frac{(a_t(\phi + \delta e_{t+1})w_t\bar{h}_t + \psi + w_t\bar{h}_t e_{t+1})}{c_t},$$

$$n_t = \frac{\gamma c_t}{(a_t(\phi + \delta e_{t+1})w_t\bar{h}_t + \psi + w_t\bar{h}_t e_{t+1})}. \tag{B.37}$$

The optimal choice of investment in child's education is given by,

$$\frac{dU_t}{de_{t+1}} = \frac{1}{c_t}\frac{dc_t}{de_{t+1}} + \frac{\gamma}{h_{t+1}}\frac{dh_{t+1}}{de_{t+1}} = 0,$$

$$\frac{\gamma\eta}{(\theta + e_{t+1})} = \frac{(w_t\bar{h}_t n_t + w_t\bar{h}_t\delta a_t n_t)}{c_t},$$

$$e_{t+1} = \frac{\eta(a_t\phi w_t\bar{h}_t + \psi) - w_t\bar{h}_t\theta(1 + \delta a_t)}{(1 - \eta)(1 + \delta a_t)w_t\bar{h}_t}. \tag{B.38}$$

Substituting (B.36),(B.37) in budget constraint,

$$c_t = \frac{a_t w_t\bar{h}_t}{(1 + \beta + \gamma)}. \tag{B.39}$$

Substituting (B.39) in (B.36),

$$s_t = \frac{\beta w_t h_t}{(1 + \beta + \gamma)}.$$

Using (B.39), (B.38) and (B.37),

$$n_t = \frac{\gamma(1 - \eta)a_t w_t\bar{h}_t}{(1 + \beta + \gamma)(a_t\phi w_t\bar{h}_t + \psi - w_t\bar{h}_t\theta(1 + \delta a_t))}.$$

We consider three important cases for which the optimal choice of fertility and investment in child's education is given by,

Case 1: $w_t\bar{h}_t < \frac{\phi\eta - \theta\delta}{\theta}\left[\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)} + \frac{\eta\psi}{\phi\eta - \theta\delta}\right]$,

$$n_t = \begin{cases}\frac{\gamma(1-\eta)a_t w_t\bar{h}_t}{(1+\beta+\gamma)(a_t\phi w_t\bar{h}_t+\psi-w_t\bar{h}_t\theta(1+\delta a_t))}, & \text{if } \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t} \le a_t,\end{cases}$$

$$e_{t+1} = \begin{cases}\frac{\eta(a_t\phi w_t\bar{h}_t+\psi)-w_t\bar{h}_t\theta(1+\delta a_t)}{(1-\eta)(1+\delta a_t)w_t\bar{h}_t}, & \text{if } \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t} \le a_t.\end{cases}$$

Knife edge case: $w_t\bar{h}_t = \frac{\phi\eta-\theta\delta}{\theta}\left[\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)} + \frac{\eta\psi}{\phi\eta-\theta\delta}\right]$,

$$n_t = \begin{cases}\frac{\gamma(1-\eta)a_t w_t\bar{h}_t}{(1+\beta+\gamma)(a_t\phi w_t\bar{h}_t+\psi-w_t\bar{h}_t\theta(1+\delta a_t))}, & \text{if } \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t} \le a_t,\end{cases}$$

$$e_{t+1} = \begin{cases}\frac{\eta(a_t\phi w_t\bar{h}_t+\psi)-w_t\bar{h}_t\theta(1+\delta a_t)}{(1-\eta)(1+\delta a_t)w_t\bar{h}_t}, & \text{if } \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t} \le a_t.\end{cases}$$

Case 2: $w_t\bar{h}_t > \frac{\phi\eta-\theta\delta}{\theta}\left[\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)} + \frac{\eta\psi}{\phi\eta-\theta\delta}\right]$,

$$n_t = \begin{cases}\frac{\gamma a_t w_t\bar{h}_t}{(1+\beta+\gamma)\left[a_t\phi w_t\bar{h}_t+\psi\right]}, & \text{if } \frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t} \le a_t < \frac{\theta}{\phi\eta-\theta\delta} - \frac{\eta\psi}{(\phi\eta-\theta\delta)w_t\bar{h}_t} \\ \frac{\gamma(1-\eta)a_t w_t\bar{h}_t}{(1+\beta+\gamma)(a_t\phi w_t\bar{h}_t+\psi-w_t\bar{h}_t\theta(1+\delta a_t))} & \text{if } \frac{\theta}{\phi\eta-\theta\delta} - \frac{\eta\psi}{(\phi\eta-\theta\delta)w_t\bar{h}_t} \le a_t,\end{cases}$$

$$e_{t+1} = \begin{cases}0, & \text{if } \frac{\bar{c}(1+\beta)}{(1+\beta-\gamma)w_t\bar{h}_t} \le a_t < \frac{\theta}{\phi\eta-\theta\delta} - \frac{\eta\psi}{(\phi\eta-\theta\delta)w_t\bar{h}_t} \\ \frac{\eta(a_t\phi w_t\bar{h}_t+\psi)-w_t\bar{h}_t\theta(1+\delta a_t)}{(1-\eta)(1+\delta a_t)w_t\bar{h}_t}, & \text{if } \frac{\theta}{\phi\eta-\theta\delta} - \frac{\eta\psi}{(\phi\eta-\theta\delta)w_t\bar{h}_t} \le a_t.\end{cases}$$

B.2 Derivation of equation (20)

We know that,

$$\kappa_{t+1} = \left(\frac{a_{t+1}}{a_t}\right)\left(\frac{n_{t-1}}{n_t}\right)\left(\frac{s_t}{s_{t-1}}\right)\left(\frac{h_t}{h_{t+1}}\right)\left(\frac{1}{1+g_{t+1}}\right)\left[\frac{a_t - n_t(a_t(\phi+\delta e_{t+1})+e_{t+1})}{a_{t+1}-n_{t+1}(a_{t+1}(\phi+\delta e_{t+2})+e_{t+2})}\right]\kappa_t,$$

Therefore, after substituting the values we derive the optimal value of κ_{t+1}

Case: $c_t + s_t = \bar{c}$ and $e_{t+1} = 0$,

$$\kappa_{t+1} = A\left[\frac{a_t^{1-\tau}(1+g_{t+1}^{\xi})\left(\bar{c}\phi+\psi\right)}{\theta^{\eta}(1+g_{t+1})(a_t z_t - \bar{c})}\right],$$

where

$$A = \left(\frac{n_{t-1}}{B}\right)\left(\frac{a_{t+1}}{a_{t+1}-n_{t+1}(\phi a_{t+1})}\right).$$

Case: $c_t + s_t = \bar{c}$ and $e_{t+1} > 0$,

$$\kappa_{t+1} = A'\left[\frac{(1+g(e_t,P_t)^{\xi})(a_t z_t\psi(1-\eta)+\bar{c}(a_t\phi z_t+\eta\psi-z_t\theta(1+\delta a_t)))}{a_t^{\tau}(1-\eta(g(e_t,P_t)))(1+g(e_t,P_t))(\theta+e_{t+1}(z_t,g_{t+1}))^{\eta(g(e_t,P_t))}z_t(a_t z_t-\bar{c})}\right],$$

where,

$$A' = \frac{n_{t-1}a_{t+1}}{B(a_{t+1}-n_{t+1}((\phi+\delta e_{t+1})a_{t+1}+e_{t+1}))}.$$

Case: $c_t + s_t > \bar{c}$, $e_{t+1} > 0$,

$$\kappa_{t+1} = B\left[\frac{(1+g_{t+1}^{\xi})\left((1+\beta)(a_t\phi z_t + \psi - z_t\theta(1+\delta a_t) + \gamma\psi(1-\eta))\right)}{a_t^{\tau}(\theta + e_{t+1})^{\eta}(1+g_{t+1})(1-\eta)}\right],$$

where

$$B = \left(\frac{\beta n_{t-1}}{B\gamma(\beta+\gamma)s_{t-1}}\right)\left(\frac{a_{t+1}}{a_{t+1} - n_{t+1}((\phi + \delta e_{t+2})a_{t+1} + e_{t+2})}\right).$$

B.3 Derivation of equation (21)

Case: $c_t + s_t = \bar{c}$,

$$e_{t+1} = 0,$$

$$e_{t+1} - e_t = 0.$$

Case: $c_t + s_t \geq \bar{c}$, $e_{t+1} > 0$,

$$e_{t+1} = \frac{\eta(g_{t+1})(\phi a_t z_t + \psi) - z_t\theta(1+\delta a_t)}{(1-\eta(g_{t+1}))z_t(1+\delta a_t)},$$

$$e_t = \frac{\eta(g_t)(\phi a_{t-1} z_{t-1} + \psi) - z_{t-1}\theta(1+\delta a_{t-1})}{(1-\eta(g_t))z_{t-1}(1+\delta a_{t-1})},$$

$$e_{t+1} - e_t = \frac{\eta(g_{t+1})(\phi a_t z_t + \psi) - z_t\theta(1+\delta a_t)}{(1-\eta(g_{t+1}))z_t(1+\delta a_t)} - \frac{\eta(g_t)(\phi z_{t-1} a_{t-1} + \psi) - z_{t-1}\theta(1+\delta a_{t-1})}{(1-\eta(g_t))z_{t-1}(1+\delta a_{t-1})}.$$

B.4 Derivation of equations (22)-(27)

For households lying within Malthusian regime, the value of κ_{t+1} assuming uniform distribution is given by,

$$K_{t+1} = P_t \int_{\frac{\bar{c}}{w_t \bar{h}_t}}^{\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t}} \frac{\beta\bar{c}}{1+\beta}\, g(a_t)da_t,$$

$$K_{t+1} = \frac{P_t(1+\beta)\beta w_t \bar{h}_t \bar{c}}{(1+\beta)(\gamma)(\bar{c})} \int da_t.$$

Substituting the limits we get,

$$K_{t+1} = \frac{P_t\beta\bar{c}}{(1+\beta)} \equiv P_t * A,$$

$$K_t = P_{t-1} * A;$$

$$\frac{K_{t+1}}{K_t} = \frac{P_t}{P_{t-1}}.$$

For households lying within **MG** regime,

$$K_{t+1} = P_t \int_{\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)w_t\bar{h}_t}}^{\frac{(w_t\bar{h}_t\theta-\psi)w_t\bar{h}_t}{\psi}} \frac{\beta w_t \bar{h}_t a_t}{1+\beta+\gamma} g(a_t) da_t,$$

$$K_{t+1} = P_t \left[\frac{\bar{c}\beta}{(1+\beta)} + \frac{\beta(w_t\bar{h}_t)^2(w_t\bar{h}_t\theta - \psi)}{(1+\beta+\gamma)\psi} \right] \equiv P_t * B(w_t\bar{h}_t).$$

For households lying within **MMG** regime,

$$K_{t+1} = P_t \int_{\frac{(w_t\bar{h}_t\theta-\psi)w_t\bar{h}_t}{\psi}}^{\frac{4(w_t\bar{h}_t\theta-\psi)w_t\bar{h}_t}{\psi}} \frac{\beta w_t \bar{h}_t a_t}{(1+\beta+\gamma)} g(a_t) da_t,$$

$$K_{t+1} = \frac{5\beta(w_t\bar{h}_t)^2(w_t\bar{h}_t\theta - \psi)P_t}{(1+\beta+\gamma)\psi} \equiv P_t * C(w_t\bar{h}_t).$$

For households lying with in Malthusian regime, we know,

$$L_t = \frac{P_t\bar{h}_t z_t(1+\beta)}{\gamma\bar{c}} \left[\int_{\frac{\bar{c}}{z_t}}^{\frac{\theta}{\phi\eta-\theta\delta}-\frac{\eta\psi}{(\phi\eta-\theta\delta)z_t}} a_t(1-\phi n_t) + \int_{\frac{\theta}{\phi\eta-\theta\delta}-\frac{\eta\psi}{(\phi\eta-\theta\delta)z_t}}^{\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)z_t}} a_t(1-(\phi+\delta e_{t+1})n_t) - e_{t+1}n_t \right] da_t,$$

$$\begin{aligned} L_t = \frac{P_t(1+\beta)}{\gamma\bar{c}w_t} & \left[\frac{\psi+\phi\bar{c}}{\phi^2} \left[\frac{\theta\phi z_t}{\phi\eta-\theta\delta} - \frac{\eta\psi\phi}{\phi\eta-\theta\delta} - \bar{c}\phi - \psi log \left[\frac{\theta\phi z_t - \psi\theta\delta}{(\phi\eta-\theta\delta)(\bar{c}\phi+\psi)} \right] \right] \right] + \frac{P_t(1+\beta)}{\gamma\bar{c}w_t} \\ & \left[\bar{c} \left[\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)} - \frac{\theta z_t}{\phi\eta-\theta\delta} + \frac{\eta\psi}{(\phi\eta-\theta\delta)} \right] \right. \\ & \left. + \frac{\psi(1-\eta)}{(\phi-\theta\delta)^2} \left[\frac{\bar{c}(1+\beta+\gamma)(\phi-\theta\delta)}{(1+\beta)} - \left[\frac{\theta z_t(\phi-\theta\delta)}{\phi\eta-\theta\delta} - \frac{(\phi-\theta\delta)\eta\psi}{\phi\eta-\theta\delta} \right] \right] \right] \\ & - \frac{P_t(1+\beta)}{\gamma\bar{c}w_t} \left[\frac{\psi(1-\eta)}{(\phi-\theta\delta)^2} \left[\bar{c}(\phi-\theta\delta) + \psi - z_t\theta \right] \right. \\ & \left. log \left[\frac{(\phi\eta-\theta\delta)\left(\bar{c}(1+\beta+\gamma)(\phi-\theta\delta) + (1+\beta)(\psi - z_t\theta)\right)}{(1+\beta)(1-\eta)\theta(\phi z_t - \delta\psi)} \right] \right]. \end{aligned}$$

Hence, we can write that

$$L_t = \frac{P_t(1+\beta)}{(\bar{c}\gamma)w_t} f\left(w_t\bar{h}_t = z_t(\kappa_t, e_t, g_t), \eta(g_{t+1})\right),$$

where,

$$\frac{dL_t}{dz_t} < 0,$$

and,

$$\frac{dL_t}{d\eta} > 0.$$

Similarly we can solve for L_t when economy lies within **MG** and **MMG** regime and calculate κ_{t+1} in all the three regimes and arrive at Equation (24). Now, calculating optimal value of e_{t+1} when economy lies within Malthusian regime,

$$e_{t+1} = \int_{\frac{\theta}{\phi\eta-\theta\delta}-\frac{\eta\psi}{(\phi\eta-\theta\delta)z_t}}^{\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)z_t}} \frac{\eta(a_t\phi w_t\bar{h}_t+\psi)-w_t\bar{h}_t\theta(1+\delta a_t)}{(1-\eta)w_t\bar{h}_t(1+\delta a_t)} da_t,$$

$$\begin{aligned} e_{t+1} = \Bigg[& \frac{1}{(1-\eta)z_t}\Bigg[\left[\frac{\eta}{\delta^2}\right]\left[\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)} - \frac{\theta z_t}{\phi\eta-\theta\delta} + \frac{\eta\psi}{\phi\eta-\theta\delta}\right] \\ & \left[\frac{(1+\beta)z_t(\phi\eta-\theta\delta)+\delta\bar{c}(1+\beta+\gamma)(\phi\eta-\theta\delta)-\theta(1+\beta)z_t+\eta\psi(1+\beta)}{(1+\beta)z_t(\phi\eta-\theta\delta)}\right. \\ & -\frac{1}{z_t}log\left[\frac{(1+\beta)z_t(\phi\eta-\theta\delta)+\delta\bar{c}(1+\beta+\gamma)(\phi\eta-\theta\delta)-\theta(1+\beta)z_t+\eta\psi(1+\beta)}{(1+\beta)z_t(\phi\eta-\theta\delta)}\right]\Bigg]\Bigg] \\ & -\frac{\theta}{2}\left[\frac{\bar{c}(1+\beta+\gamma)}{(1+\beta)} + \frac{\theta z_t}{\phi\eta-\theta\delta} - \frac{\eta\psi}{(\phi\eta-\theta\delta)}\right]\Bigg] \\ & \equiv \Phi^a(e_t, g_t, \kappa_t; P). \end{aligned}$$

Similarly we can solve for an economy lying within **MG** and **MMG** regime and calculate e_{t+1} in all three regimes and arrive at Equation (25).

B.5 Income-fertility path of 21 OECD countries

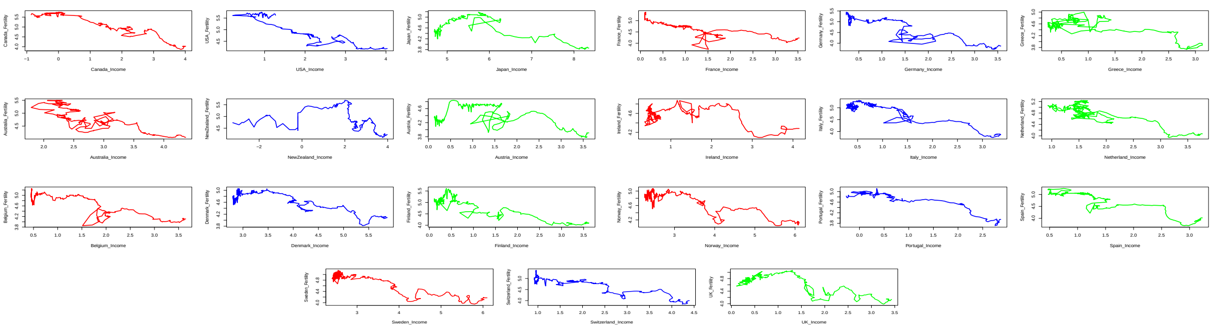


Figure 14: Income-fertility path of 21 OECD countries

B.6 Historical fertility path (over time) of 21 OECD countries

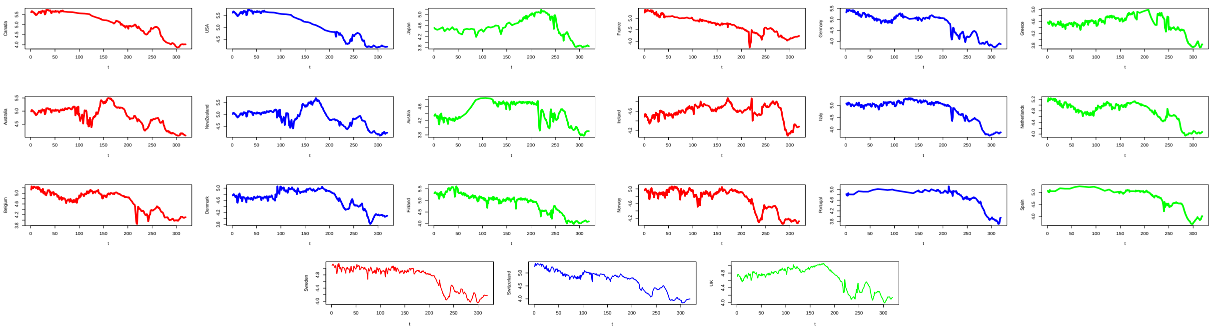


Figure 15: Historical fertility path (over time) of 21 OECD countries.

B.7 Rising return to education with technological progress

We describe how returns to education varies with technological progress. To calculate Mincer returns to education, we use the Penn World Table (PWT) 10.1 and the Barro and Lee (2018) dataset. The formula used to compute the Mincer returns to education is as follows:

$$\text{human capital} = e^{\text{Mincer returns to education * year of schooling}},$$

$$\text{Mincer returns to education} = \frac{log(\text{human capital})}{\text{year of schooling}}.$$

The data on years of schooling is taken from the Barro and Lee (2018) dataset, while human capital is obtained from the Penn World Table (PWT) 10.1 to calculate the Mincer returns to education. Information on total factor productivity (TFP) is also taken from the PWT. Specifically, the PWT provides the data on variable *ctfp*, which measures TFP at current purchasing power parity (PPP) prices. Furthermore, we compute Solow residual TFP growth using the PWT variables *rgdpna*, *rnna*, *emp*, *avh*, and *labsh*, which represent real GDP at constant 2017 national prices, capital stock at constant prices, number of persons employed, average annual hours worked per worker, and the labor share of income, respectively. Importantly, we use total hours worked (derived from $emp \times avh$) rather than employment alone to account for variations in hours worked. The Solow residual TFP growth can then be calculated from PWT using the following standard growth accounting formula:

$$TFP\ growth = \Delta ln(Y) - \alpha \Delta ln(K) - (1 - \alpha)\Delta ln(L),$$

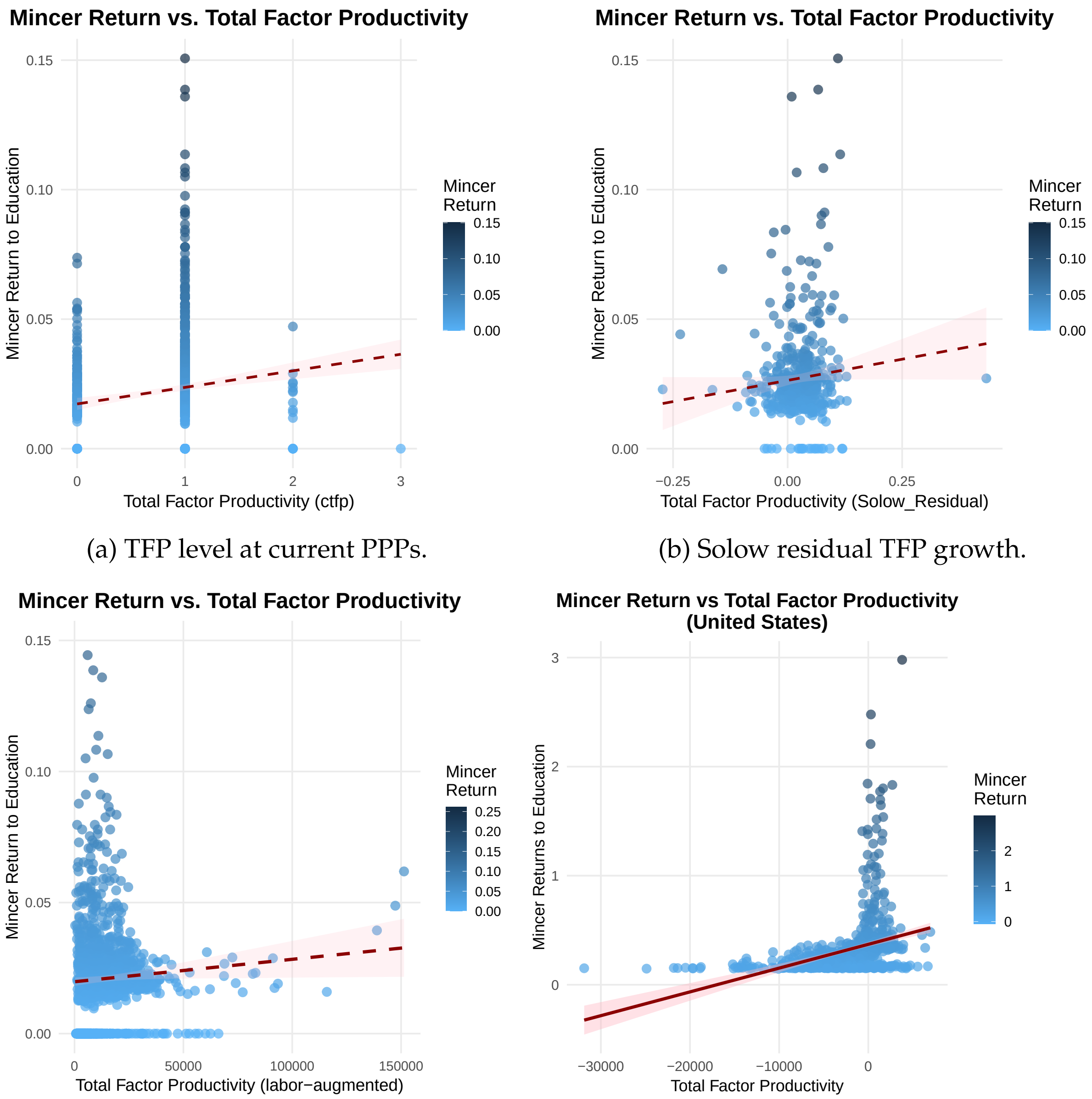
where Y is real GDP at constant prices (*rgdpna* in PWT), K is capital stock (*rnna* in PWT), L is labor input (Total hours worked, $avh * emp$ in PWT) and α is capital share of income (from PWT: *labsh* is the labor share, so $\alpha = 1 - labsh$). In addition, we calculate labor-augmented TFP growth using PWT variables. The formula for labor-augmented TFP (also referred to as TFP per effective worker) is given by:

$$TFP\ growth = \frac{Y}{K^{\alpha}(h.L)^{1-\alpha}},$$

where Y is output-side real GDP at constant 2017 national prices (*rgdpo*), K is capital stock (*rnna* in PWT), h is human capital index (*hc* in PWT), L is employment (*emp* in PWT) and α is capital share. Figure 16a, 16b, and 16c show a positive relationship between total factor productivity and the Mincer returns to education where TFP is measured using *ctfp*, Solow residual TFP growth and labor-augmented TFP growth using PWT 10.1 and Barro and lee (2018) dataset which is a cross country analysis. In contrast, Figure 16d indicates a positive relationship between the Mincer returns to education and TFP for USA using Turner et al. (2013) dataset. The dataset cover 28 states of USA for a period of 1840-2000. They have estimates for years of schooling and also constructed original per worker estimates of human capital, physical capital and output. They then used growth accounting technique to estimate TFP to output growth. The formula used to calculate Mincer returns to education follows the same approach as in PWT 10.1. Total Factor Productivity (TFP) growth is calculated using the following expression:

$$TFPgrowth = y - \alpha k - (1 - \alpha)h$$

where y is output per worker, k is capital per worker and h is human capital per worker.



(a) TFP level at current PPPs.

(b) Solow residual TFP growth.

(c) Labor-augmented TFP growth.

(d) TFP (Turner et al. (2013)).

Figure 16: Mincer Return vs. Total factor Productivity.